A Unified Approach to Estimating Demand and Welfare Changes*

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Abstract

The measurement of price changes, economic welfare, aggregate parameters, and demand patterns is currently based on three disjoint approaches: macroeconomic models derived from time-invariant utility functions, microeconomic estimation based on time-varying utility (demand) systems, and actual price and real output data constructed using formulas that differ from either approach. The inconsistencies are so deep that the same assumptions that form the foundation of demand-system estimation can be used to prove that standard price indexes are incorrect, and the assumptions underlying standard exact and superlative price indexes invalidate demand-system estimation. In other words, we show that extant micro and macro welfare estimates are mutually inconsistent, and neither is consistent with the data. We develop a unified approach to demand and price measurement that exactly rationalizes observed micro data on prices and expenditure shares while permitting exact aggregation and meaningful macro comparisons of welfare over time. We show that all standard price indexes are special cases of our approach for particular values of the elasticity of substitution and a constant set of goods. In contrast to these standard index numbers, our approach allows us to compute changes in the cost of living that take into account both changes in the demand for individual goods and the entry and exit of goods over time. Using barcode data for the U.S. consumer goods industry, we show that allowing for the entry and exit of products, changes in demand for individual goods, and a value for the elasticity of substitution estimated from the data yields substantially different conclusions for changes in the cost of living from standard index numbers

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1 Introduction

The measurement of economic welfare, aggregate parameters, and demand patterns is currently based on three disjoint approaches: macroeconomic models derived from *time-invariant* utility functions, microeconomic estimation based on *time-varying* utility (demand) systems, and actual price and real output data constructed using formulas that differ from either approach. The inconsistencies are so deep that the same assumptions that form the foundation of demand-system estimation can be used to prove that standard price indexes are incorrect, and the assumptions underlying standard exact and superlative price indexes invalidate demand-system estimation. In other words, we show that extant micro and macro welfare estimates are mutually inconsistent, and neither is consistent with the data.

In order to deal with this problem, our paper presents a new empirical methodology, which we term "the unified approach," that reconciles all major micro, macro, and statistical approaches. Our "unified price index" nests all major price indexes used in welfare or demand system analysis. Thus, how economists and statistical agencies currently measure welfare can be understood in terms of an internally consistent approach that has been altered by ignoring data, moment conditions, and/or imposing particular parameter restrictions. For example, the U.S. bases its consumption prices on the Jevons (1865) index. The Bureau of Labor Statistics's usage of sales-frequency sampling corrects for the fact that goods are purchased at different frequencies and thereby converts the Jevons into a Cobb-Douglas index. Allowing the elasticity of substitution to differ from the Cobb-Douglas assumption of one produces the Sato-Vartia (1976) CES exact price index. Introducing the entry and exit of goods over time generates the Feenstra-CES index (Feenstra (1994). Incorporating demand shocks for each good and estimating the elasticity of substitution using the assumption of a constant aggregate utility function produces the unified index. Other paths are shorter. The unified index exactly corresponds to expected utility if consumers have heterogeneous random utility with extreme value distributions (e.g., Logit or Fréchet). Similarly, the Dutot (1738), Carli (1764), Laspeyres (1871) and Paasche (1875) indexes all can be derived from the unified approach by making the appropriate parameter restrictions. Finally, relaxing assumptions necessary to yield the Fisher (1922) and Törnqvist (1936) indexes, yields the broader class of quadratic mean price indexes. The Sato-Vartia index arises naturally in this class, and as we just discussed, yields the unified price index if it is generalized. In other words, many seemingly fundamentally different approaches to welfare measurement—*e.g.*, Laspeyres and Cobb-Douglas indexes—are actually linked together via the unified approach.

The first key insight of the unified approach is that any demand system errors (e.g., taste shocks) must show up in the utility and unit expenditure functions, and therefore the price index. However, all extant exact and superlative indexes (such as the Sato-Vartia, Fisher and the Törnqvist) are derived under the assumption that product demand is *time invariant*. Researchers made this assumption because it is a *sufficient* condition to guarantee a constant utility function. Unfortunately, this assumption creates a conundrum. As we show in the paper, if one assumes that demand shocks are time invariant, one can solve for the elasticity of substitution *without doing estimation!* Thus, if one believes the assumption underlying all economically motivated price indexes—that demand does not shift—demand system estimation is both wrong (because it assumes demand shifts) and irrelevant (because identification does not require econometrics). Alternatively, if one believes the overwhelming evidence that demand is not constant over time, then this violates the assumptions underlying economic approaches macro price and welfare measurement. In other words, macro and micro approaches are based on contradictory assumptions: either one can believe the assumption of constant demand underlying exact price indexes, which means that demand-system estimates are incorrect, or one can believe micro evidence that demand shifts, which means our price and real output measures are incorrect.

The solution to bridging the micro-macro divide requires our second key insight, which is to show that the assumption of time-invariant demand for each good is neither the correct nor the necessary condition for constant aggregate utility in the presence of time-varying demand shocks for each goods. One of the most surprising results from incorporating demand shocks into the utility function is that identification of key parameters is vastly simpler if one uses both the information in the expenditure function and demand system. For example, we show that in the CES setup with continuous and differentiable prices and expenditure shares, a researcher knowing only prices and expenditure shares can solve for the elasticity of substitution without knowing *anything* about supply conditions. The intuition arises from simply counting equations and unknowns. If we think about a dataset containing price and share changes for k goods, we have $k + 1$ unknowns (one unknown price elasticity and k unknown values for the product appeal changes). However, we also have a system of $k + 1$ equations (one demand equation for each of the k goods and one equation for the change in the unit expenditure function). The system is exactly identified. In other words, given data on prices and expenditure shares and the assumption of a constant aggregate utility function, one does not need to estimate demand parameters; one just solves for them.

As we show, the problem is more complex when there are discrete changes because price and expenditure share derivatives become discrete differences, but the same basic intuition applies. With discrete changes, we show that there are two ways of writing the change in the unit expenditure function: one using the preference weights of consumers in the start period and the second using the weights of consumers in the end period. We develop a "reverse-weighting" estimator that identifies the elasticity that brings these two ways of writing the change in prices as close together as possible and demonstrate that it consistently estimates the correct elasticity theoretically and in Monte Carlo simulations. This reverse-weighting elasticity enables us to write down the unified price index. Our index is exact for the CES utility function under the assumptions of time-varying and constant demand as well as when the set of goods is changing. Moreover, it exhibits other good "axiomatic" properties that many other indexes fail to satisfy, such as transitivity and time reversibility.

We focus on the CES functional form, because there is little doubt that this is the preferred approach to modeling product variety across international trade, economic geography and macroeconomics, but we consider several potential shortcomings. Our CES price index is not superlative, because it does not approximate any continuous and differentiable utility function. But superlative indexes like Fisher (used in the personal consumption expenditure index) and Törnqvist are closely related to CES indexes because they arise from quadratic mean utility functions. Indeed, we find that if we impose similar parameter restrictions on our unified price index (no demand shocks or variety changes), the differences in measured price changes between our index and superlative indexes are trivial. This result establishes that the key differences between the unified and superlative indexes stem from assumptions about the existence of demand shocks or new goods, not functional forms.

A second concern arises from the concern that agents may not be homogeneous. Our unified index features symmetry and homotheticity and exhibits an independence of irrelevant alternatives (IIA) property (the relative expenditure of any two varieties only depends on the characteristics of those varieties and not on the characteristics of other varieties within a market). Building price indexes when this assumption is violated has proven to be a vexing issue for economists. For example, Deaton (1998) writes, "it is unclear that a quality-corrected cost-of-living index in a world with many heterogenous agents is an operational concept." In order to address this concern, we show, as an extension, how to break these features by allowing for heterogeneous consumers with different elasticities of substitution and product appeal. Thereby, unifying the heterogeneous consumer and price index literatures.

Our paper is related to a number of strands of existing research. First, we build on a long line of existing research on price indexes. Existing price measurement is based on the "statistical approach" to price indexes developed by Dutot (1738), Carli (1764), and Jevons (1865). The methodologies developed in these papers forms the foundation of 98 percent of all consumer price indexes generated by government statistical agencies (Stoevska 2008). We show how sampling techniques convert these indexes into Laspeyres (1871), Paasche (1875), and Cobb-Douglas price indexes.[1] These indexes are in turn closely related to our unified price index as well as the Fisher (1922) and Törnqvist (1936) price indexes.

However, the path to the unified price index need not start with the actual price indexes used by statistical agencies. Following Konüs (1924), economic theory has largely rejected the "statistical approach" to price measurement in favor of the "economic approach," which asserts that all price indexes should be derived from consumer theory and correspond to the unit expenditure function. The subsequent economic approach to price measurement, including Diewert (1976), Sato (1976), Vartia (1976), Lau (1979), Moulton (1996), Balk (1999) and Feenstra and Reinsdorf (2010), has focused on exact and superlative index numbers that feature time-invariant demand parameters. Our unified price index also arises naturally when following this economic approach. We show how to relax the assumption of time invariant demand for each good while preserving a constant aggregate utility function to make welfare comparisons over time. Thus, although there has been an international rift in the approach to measuring inflation—with the U.S. Department of Labor accepting the economic approach to price measurement and U.K. statistical agencies explicitly rejecting it (Triplett 2001)—we show these debates can be reduced to asking what restrictions should be placed on the unified approach.

It is an interesting feature of the literature that even path-breaking economists who have taught us how to measure price index parameters using time-varying demand often assume it to be false in the same work when they measure prices and welfare. For example, Deaton and Muellbauer (1980) provide extensive discussions of time-varying demand in the estimation of the demand system. However, when they use unit expenditure functions that are standard in the price index literature in order to show how to measure welfare changes, there is no discussion of the fact that these were derived (elsewhere) based

[1] The "Cobb-Douglas" functional form was first used by Wicksell (1898) and the price index was discovered by Konyus (Konüs) and Byushgens (1926). Cobb and Douglas (1928) applied it to U.S. data.

on a time-invariant formulation of demand. Similarly, Feenstra (1994), identifies CES parameters based on heteroskedasticity of demand shocks, and explicitly acknowledges the inconsistency between the demand system estimation and the CES price index, but does not resolve it.

Our results suggest that the measurement of price changes is far more model dependent than economists typically believe. While the differences among conventional price indexes—whether economically or statistically motivated—tend to be small, we can show that this is almost entirely driven by the fact that these indexes are all built under the assumption that there are no time-varying demand shifts. The moment one allows for time-varying demand the correlation between conventional price indexes and our unified price index plummets to almost zero when one considers individual product groups and only 0.5 when constructing aggregate measures. Intuitively, these demand shifts tend to cancel as one aggregates across time, so the choice of index matters less as one considers longer differences. However, even this result is crucially dependent on whether one compares indexes that adjust for new goods. Indexes that do not do this, have severe upward biases.

Our study is also related to a more recent, voluminous literature in macroeconomics, trade and economic geography that has used CES preferences. This literature includes, among many others, Anderson and van Wincoop (2003), Antràs (2003), Arkolakis, Costinot and Rodriguez-Clare (2012), Armington (1969), Bernard, Redding and Schott (2007, 2011), Blanchard and Kiyotaki (1987), Broda and Weinstein (2006, 2010), Dixit and Stiglitz (1977), Eaton and Kortum (2002), Feenstra (1994), Helpman, Melitz and Yeaple (2004), Hsieh and Klenow (2009), Krugman (1980, 1991), Krugman and Venables (1995) and Melitz (2003). We provide a new approach for determining the elasticity of substitution from observed data on prices and expenditures. When product appeal is constant over time, the elasticity of substitution can be recovered from the observed data with no estimation required. When product appeal is changing over time, the estimation problem has a generalized method of moments (GMM) representation and the resulting estimates satisfy the joint implications of duality, time reversibility and aggregation.

Our work is also related work in macroeconomics aimed at measuring inflation, real output, and quality change. Shapiro and Wilcox (1996) sought to back out the elasticity of substitution in the CES index by equating it to a superlative index. Bils and Klenow (2001) quantified quality growth in U.S. prices; our work provides a general framework for thinking about these types of changes.

Finally, our analysis connects with the broader literature on demand systems estimation, including McFadden (1974), Deaton and Muellbauer (1980), Anderson, de Palma and Thisse (1992), Berry (1994), Berry, Levinsohn and Pakes (1995), McFadden and Train (2000) and Sheu (2014). A related literature examines the implications of new goods for welfare, including Feenstra (1994), Bresnahan and Gordon (1996), Hausman (1996), Broda and Weinstein (2006, 2010) and Petrin (2002) In contrast to these literatures, our method emphasizes the intimate relationship between price indexes and demand systems. We provide an approach that exactly rationalizes the observed data on prices and expenditure as an equilibrium while also satisfying duality, time reversibility and aggregation, and hence permitting meaningful comparisons of aggregate welfare over time.

The remainder of the paper is structured as follows. Section 2 develops our theoretical framework and the main theoretical propositions in the paper. Section 3 shows the relationships between our unified price index and standard price indexes used by economists and statistical agencies. Section 4 shows how to incorporate heterogeneous groups of consumers with different substitution parameters. Section 5 reports our empirical results, including a Monte Carlo simulation to illustrate the performance of our estimator, and an application to barcode data for the U.S. consumer goods sector. Section 6 concludes.

2 Model

Our first step towards solving the problem of how to unify welfare analysis and demand-system estimation is be explicit about the challenge we face. We will work with a CES utility function, but we will generalize the setup into a world of multiple groups of consumers with heterogeneous substitution and demand parameters (which allows for non-homothetic preferences across groups with different levels of income). We begin by considering a CES utility function with time-varying demand parameters and writing down the price index and demand system that is compatible with it. We next derive the variety-adjusted, exact CES price index for a world in which demand for each good is time varying. Our second step is to understand what it means to have constant aggregate utility in a world with time-varying demand for individual goods. In order to facilitate the reader's intuition for the method, we begin by considering the case in which prices and expenditure shares are continuous and differentiable, before allowing for discrete changes in prices and expenditure shares.

2.1 Duality

Utility (U_t) is defined over the consumption (C_{kt}) of each good k at time t:

$$\mathbb{U}_t = \left[\sum_{k \in \Omega_t} \left(\varphi_{kt} C_{kt} \right)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}, \qquad \sigma \in (-\infty, \infty), \qquad \varphi_{kt} > 0, \tag{1}$$

where σ is the elasticity of substitution across goods; φ_{kt} is the preference ("demand") parameter for good k at time t; the set of goods at time t is denoted by Ω_t; and we indicate the number of elements in this set by $N_t = |\Omega_t|$. Although we allow demand parameters for individual goods (φ_{kt}) to change over time, we continue to assume a constant aggregate utility function to permit meaningful comparisons of welfare over time, which implies a constant elasticity of substitution (σ) over time. The corresponding dual price index ($\mathbb{P}_t$) (or unit expenditure function) is defined over the price (P_{kt}) of each good k at time t:

$$\mathbb{P}_t = \left[\sum_{k \in \Omega_t} \left(\frac{P_{kt}}{\varphi_{kt}} \right)^{1-\sigma} \right]^{\frac{1}{1-\sigma}}. \tag{2}$$

Our goal is to estimate the elasticity of substitution (σ) and demand (φ_{kt}) for each good $k \in \Omega_t$ and period t, where the demand parameters (φ_{kt}) correspond to structural residuals that ensure that the model exactly replicates the observed data on prices and expenditure shares for each good and period.

The CES utility function (1) and (2) implies a demand system in which expenditure shares are

$$S_{\ell t} \equiv \frac{P_{\ell t} C_{\ell t}}{\sum_{k \in \Omega_t} P_{kt} C_{kt}} = \frac{\left(P_{\ell t} / \varphi_{\ell t} \right)^{1-\sigma}}{\sum_{k \in \Omega_t} \left(P_{kt} / \varphi_{kt} \right)^{1-\sigma}}, \qquad k, \ell \in \Omega_t. \tag{3}$$

We allow for the entry and exit of goods over time. In particular, we partition the set of goods in period t (Ω_t) into those "common" to t and $t-1$ ($\Omega_{t,t-1}$) and those added between $t-1$ and t (I_t^+), where $\Omega_t = \left\{\Omega_{t,t-1} \cup I_t^+\right\}$. Similarly, we partition the set of goods in period $t-1$ (Ω_{t-1}) into those common to t and $t-1$ ($\Omega_{t,t-1}$) and those dropped between $t-1$ and t (I_t^-), where $\Omega_{t-1} = \left\{\Omega_{t,t-1} \cup I_{t-1}^-\right\}$. We denote the number of goods in period t by N_t and the number of common goods by $N_{t,t-1} = |\Omega_{t,t-1}|$. We assume that $\varphi_{ut} = 0$ for a good u before it enters and after it exits, which rationalizes the observed entry and exit of goods over time.

2.2 The CES Price Index with Time-Varying Demand

Our ultimate goal is to develop an estimation procedure for understanding how to estimate parameters and measure welfare when there are discrete price and expenditure movements. In order to understand the intuition for how the results are obtained, it is useful to first consider a simple case of how to analyze welfare when all variables are continuous and differentiable. We next generalize the analysis to allow for discrete changes in variables.

2.2.1 The Unified Approach to Elasticity Measurement Assuming Differentiability

Although we will relax this assumption later, we will start by analyzing a simple case in which we assume that the set of goods is not changing, *i.e.*, $\Omega_t = \Omega$ and $N_t = N$ for all t. If we take logs of both sides of our demand equation (3), we obtain the demand system that is forms the basis of econometric exercises to estimate the elasticity of substitution:

$$\ln S_{\ell t} = (1-\sigma)\ln P_{\ell t} - (1-\sigma)\ln \mathbb{P}_t - (1-\sigma)\ln \varphi_{\ell t}. \tag{4}$$

The standard econometric technique employed to estimate σ is to first difference equation (4), include a time varying fixed effect to eliminate the first-differenced price index term ($(1-\sigma)\Delta \ln \mathbb{P}_t$), and estimate (4) using an instrument ($\ln Z_{kt}$) that is correlated with changes in prices ($\Delta \ln P_{kt}$) but uncorrelated with changes in demand ($\Delta \ln \varphi_{kt}$): $\mathbb{E}\left[\Delta \ln \varphi_{kt} \ln Z_{kt} = 0\right]$.

We can use our unit expenditure function and demand system to place *two* constraints on the structure of the preference shocks. We start with the demand system given by equation (4). If we divide both sides of equation (4) by the number of goods (N), sum over all goods and exponentiate, we obtain

$$\mathbb{P}_t = \frac{\tilde{P}_t}{\tilde{\varphi}_t}\left(\tilde{S}_t\right)^{\frac{1}{\sigma-1}}, \tag{5}$$

where a tilde over a variable indicates a geometric average, *i.e.*, $\tilde{x}_t = \left(\prod_{k\in\Omega} x_{kt}\right)^{1/N}$. Totally differentiating this expression yields

$$\frac{d\mathbb{P}_t}{\mathbb{P}_t} = \frac{d\tilde{P}_t}{\tilde{P}_t} + \frac{1}{\sigma-1}\frac{d\tilde{S}_t}{\tilde{S}_t} - \frac{d\tilde{\varphi}_t}{\tilde{\varphi}_t}. \tag{6}$$

Our demand system (3) must be homogeneous of degree zero in the demand parameters (φ_{kt}). Therefore, given data on expenditure shares and prices, the demand parameters can only be identified up to a normalization (a choice of units in which to measure the demand parameters). We can therefore impose the normalization that the geometric mean of the demand parameters is is equal to one: $\tilde{\varphi}_t = \left(\prod_{k\in\Omega} \varphi_{kt}\right)^{1/N} = 1$, which guarantees that following condition (20) holds

$$\frac{d\tilde{\varphi}_t}{\tilde{\varphi}_t} = 0. \tag{7}$$

Another way of stating the same condition is that constant aggregate utility implies that the product demand shocks cannot change the price index in equation (5), which motivates the familiar econometric condition that the demand shocks must be mean zero (*i.e.*, $\mathbb{E}\left(\Delta \ln \varphi_{kt}\right) = 0$).

Our second key insight comes from realizing that changes in the price index can not only be recovered from the demand system as in equation (6), but can also be directly derived from the price index or unit expenditure function. If we totally differentiate equation (2), we have:

$$\frac{d\mathbb{P}_t}{\mathbb{P}_t} = \sum_{k\in\Omega} S_{kt}\frac{dP_{kt}}{P_{kt}} - \sum_{k\in\Omega} S_{kt}\frac{d\varphi_{kt}}{\varphi_{kt}}. \tag{8}$$

The first term in equation (8) is completely conventional: the the price index is equal to the share-weighted changes in each price. The second term is the analog for demand and follows directly from the fact that the price index must be homogeneous of degree 1 in prices. Critically, however, the assumption that aggregate utility is constant requires the following additional condition to hold in our *price* index:

$$\sum_{k\in\Omega} S_{kt}\frac{d\varphi_{kt}}{\varphi_{kt}} = 0. \tag{9}$$

Thus, the CES utility function places two restrictions on demand shocks: one from the unit expenditure function and one from the demand system. Computing price changes based on the demand system requires that an *unweighted* sum of these preference shocks sum to zero, and the CES unit expenditure function ("price index") implies that a *weighted* sum of these residuals must also equal zero.

Equation (9) is not usually imposed in empirical studies, but once we realize that it must hold too, the identification problem becomes trivial. Dividing the expenditure share (3) by its geometric mean expenditure share, we obtain:

$$\varphi_{\ell t} = \frac{P_{\ell t}}{\tilde{P}_t}\left(\frac{S_{\ell t}}{\tilde{S}_t}\right)^{\frac{1}{\sigma-1}}, \tag{10}$$

where the tilde above a variable again denotes a geometric mean. Totally differentiating (10), and substituting for $d\varphi_{kt}/\varphi_{kt}$ in (9), we obtain:

$$\sum_{k\in\Omega} S_{kt}\left[\frac{dP_{kt}}{P_{kt}} - \frac{d\tilde{P}_t}{\tilde{P}_t} + \frac{1}{\sigma-1}\left(\frac{dS_{kt}}{S_{kt}} - \frac{d\tilde{S}_t}{\tilde{S}_t}\right)\right] = 0. \tag{11}$$

This equation yields a closed-form solution for σ, which we present as the following proposition.

Proposition 1. *Given continuous and differentiable values for prices and expenditure shares $\{P_{kt}, S_{kt}\}$ for a constant set of goods Ω, the assumption of constant aggregate CES preferences (equations (2), (3), (7) and (9)) identifies the elasticity of substitution ($\hat{\sigma}$) and the demand parameter ($\hat{\varphi}_{kt}$) for each good k and time period t:*

$$\hat{\sigma} = 1 + \frac{\sum_{k\in\Omega} S_{kt}\left(\frac{dS_{kt}}{S_{kt}} - \frac{d\tilde{S}_t}{\tilde{S}_t}\right)}{\sum_{k\in\Omega} S_{kt}\left(\frac{d\tilde{P}_t}{\tilde{P}_t} - \frac{dP_{kt}}{P_{kt}}\right)}, \tag{12}$$

$$\hat{\varphi}_{kt} = \frac{P_{kt}}{\tilde{P}_t}\left(\frac{S_{kt}}{\tilde{S}_t}\right)^{\frac{1}{\hat{\sigma}-1}}.$$

Proof. The proposition follows immediately from equations (11) and (3) and our normalization $\tilde{\varphi}_t = 1$. □

The intuition for why we can solve for the elasticity of substitution can be garnered by simply counting equations and unknowns. If we think about a dataset containing price and share changes, we know that we $k+1$ unknowns (one unknown value of σ and k unknown values for each possible ratio of appeal parameters of $\varphi_{kt}/\varphi_{k,t-1}$). If we express the CES demand system given by equation (4) as a time difference, we obtain k equations. The CES price index (2) provides the last equation, giving us a system of $k+1$ equations and $k+1$ unknowns. The system is exactly identified, and so we can obtain an exact price index.

Equation (12) tells us that in continuous time, the elasticity of substitution of a CES demand system can be identified without using any information about firm pricing behavior. From (12), the larger the changes in relative expenditure shares ($S_{kt}/\tilde{S}_t$) compared to the changes in relative prices ($P_{kt}/\tilde{P}_t$), the higher the implied elasticity of substitution (σ). Moreover, once we know the elasticity (σ), we can recover the unobserved time-varying appeal (φ_{kt}) from expenditure shares (3), and hence can solve for the price index ($\mathbb{P}_t$) from (2).

The elasticity of substitution σ is identified in (12) solely from the assumption of *constant aggregate CES preferences*, which imposes the restriction (7) on the mean change in appeal (φ_{kt}) and the orthogonality condition (9) on the relationship between changes in appeal and initial expenditure shares. Furthermore, the use of any other orthogonality condition to identify σ is inconsistent with the assumption of constant aggregate CES preferences, unless it also happens to satisfy the orthogonality condition (9).

2.2.2 The Unified Approach for Discrete Changes

The foregoing discussion made a number of simplifications to keep the algebra simple and provide the intuition for how the unit expenditure function and the demand system place two different constraints on demand shocks. Economists are typically interested in computing elasticities in which the set of goods is changing and when changes in prices and expenditure shares are discrete (for example, when doing comparative static exercises). This introduces a number of complexities principally because we cannot take time derivatives.

We begin by writing change in the cost of living from $t-1$ to t as the ratio between the price indexes (2) in each period:

$$\Phi_{t-1,t} = \frac{\mathbb{P}_t}{\mathbb{P}_{t-1}} = \left[\frac{\sum_{k\in\Omega_t} \left(P_{kt}/\varphi_{kt}\right)^{1-\sigma}}{\sum_{k\in\Omega_{t-1}} \left(P_{kt-1}/\varphi_{kt-1}\right)^{1-\sigma}} \right]^{\frac{1}{1-\sigma}}. \tag{13}$$

The fact that the set of goods is changing means that the set of goods in the denominator is not the same as that in in the numerator. Feenstra (1994) showed that one way around this problem is to express the price index in terms of price index for "common goods" (*i.e.*, goods available in both time periods) and a variety-adjustment term. Summing equation (3) over the set of commonly available goods, we can express expenditure on all common goods as a share of total expenditure in periods t and $t-1$ as:

$$\lambda_t \equiv \frac{\sum_{k\in\Omega_{t,t-1}} \left(P_{kt}/\varphi_{kt}\right)^{1-\sigma}}{\sum_{k\in\Omega_t} \left(P_{kt}/\varphi_{kt}\right)^{1-\sigma}}, \qquad \lambda_{t-1} \equiv \frac{\sum_{k\in\Omega_{t,t-1}} \left(P_{kt-1}/\varphi_{kt-1}\right)^{1-\sigma}}{\sum_{k\in\Omega_{t-1}} \left(P_{kt-1}/\varphi_{kt-1}\right)^{1-\sigma}}, \tag{14}$$

where λ_t is equal to the total sales of continuing goods in period t divided by the sales of all goods available in time t evaluated at *current* prices. Its maximum value is one if no goods enter in period t and will *fall* as the share of *new* goods *rises*. Similarly, λ_{t-1} is equal to total sales of continuing goods as share of total sales of all goods in the *past* period evaluated at $t-1$ prices. It will equal one if no goods cease being sold and will *fall* as the share of *exiting* goods *rises*.

Multiplying the numerator and denominator of (13) by the summation $\sum_{k\in\Omega_{t,t-1}}\left(P_{kt}/\varphi_{kt}\right)^{1-\sigma}$ over common goods at time t, and substituting in λ_t from above, we obtain:

$$\Phi_t = \left[\frac{1}{\lambda_t}\frac{\sum_{k\in\Omega_{t,t-1}}\left(P_{kt}/\varphi_{kt}\right)^{1-\sigma}}{\sum_{k\in\Omega_{t-1}}\left(P_{kt-1}/\varphi_{kt-1}\right)^{1-\sigma}}\right]^{\frac{1}{1-\sigma}}.$$

Multiplying the numerator and denominator by the summation $\sum_{k\in\Omega_{t,t-1}}\left(P_{kt-1}/\varphi_{kt-1}\right)^{1-\sigma}$ over common goods at time $t-1$, and using the variety term in period $t-1$ (14), we obtain the exact CES price index:

$$\Phi_t = \left[\frac{\lambda_{t-1}}{\lambda_t}\frac{\sum_{k\in\Omega_{t,t-1}}\left(P_{kt}/\varphi_{kt}\right)^{1-\sigma}}{\sum_{k\in\Omega_{t,t-1}}\left(P_{kt-1}/\varphi_{kt-1}\right)^{1-\sigma}}\right]^{\frac{1}{1-\sigma}} = \left(\frac{\lambda_t}{\lambda_{t-1}}\right)^{\frac{1}{\sigma-1}}\frac{\mathbb{P}_t^*}{\mathbb{P}_{t-1}^*}, \tag{15}$$

where $\mathbb{P}_t^*$ denotes the price index defined over *common* goods (*i.e.*, goods available in periods t and $t-1$):

$$\mathbb{P}_t^* \equiv \left[\sum_{k\in\Omega_{t,t-1}}\left(\frac{P_{kt}}{\varphi_{kt}}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}. \tag{16}$$

This common goods price index is the dual to the utility function if the set of goods is not changing, and it will prove to be a useful building block in our more general price indexes. The term multiplying it in equation (15) is the "variety-adjustment." The term in parentheses adjusts the index for entering and exiting goods. If new goods are more numerous than exiting goods or have lower average price per unit quality, then $\lambda_t/\lambda_{t-1} < 1$, and the price index ($\Phi_t$) will fall due to increase in variety or entering varieties having higher demand than exiting varieties.

From the CES demand structure, the share of each common good in expenditure on all common goods ($S_{\ell t}^*$) is

$$S_{\ell t}^* \equiv \frac{P_{\ell t}C_{\ell t}}{\sum_{k\in\Omega_{t,t-1}}P_{kt}C_{kt}} = \frac{\left(P_{\ell t}/\varphi_{\ell t}\right)^{1-\sigma}}{\sum_{k\in\Omega_{t,t-1}}\left(P_{kt}/\varphi_{kt}\right)^{1-\sigma}}. \tag{17}$$

If we rearrange terms we obtain the following useful relationship:

$$\left(\mathbb{P}_t^*\right)^{1-\sigma} = \sum_{k\in\Omega_{t,t-1}}\left(P_{kt}/\varphi_{kt}\right)^{1-\sigma} = \frac{1}{S_{\ell t}^*}\left(P_{\ell t}/\varphi_{\ell t}\right)^{1-\sigma}, \qquad k,\ell\in\Omega_{t,t-1}. \tag{18}$$

If we take the logs of both sides of equation (18), sum across all $k\in\Omega_{t,t-1}$, and divide both sides by the number of common goods, we find that the log change in the common goods price index also can be written as the discrete-time analog of equation (6):

$$\ln\left(\frac{\mathbb{P}_t^*}{\mathbb{P}_{t-1}^*}\right) = \ln\left(\frac{\tilde{P}_t}{\tilde{P}_{t-1}}\right) + \frac{1}{\sigma-1}\ln\left(\frac{\tilde{S}_t^*}{\tilde{S}_{t-1}^*}\right) - \ln\left(\frac{\tilde{\varphi}_t^*}{\tilde{\varphi}_{t-1}^*}\right), \tag{19}$$

where a tilde over a variable again denotes a geometric average and the asterisk indicates that the geometric average is taken for the set of common goods: $\tilde{x}_t^* = \left(\prod_{k\in\Omega_{t,t-1}}x_{kt}\right)^{1/N_{t,t-1}}$. If we impose the assumption of constant aggregate utility, which implies that the product demand shocks cannot change the aggregate common goods price index (19), the discrete-time analog of equation (7) must also hold:

$$\ln\left(\frac{\tilde{\varphi}_t^*}{\tilde{\varphi}_{t-1}^*}\right)=0, \tag{20}$$

which corresponds to the econometric assumption that the demand shocks are mean zero for common goods (*i.e.*, $\mathbb{E}\left(\Delta \ln \varphi_{kt}\right)=0$).

Substituting equations (20) and (19) into (17) produces the "demand-based" change in the cost of living going forward in time from $t-1$ to t :

$$\Phi_{t-1,t}^D=\left(\frac{\lambda_t}{\lambda_{t-1}}\right)^{\frac{1}{\sigma-1}}\frac{\mathbb{P}_t^*}{\mathbb{P}_{t-1}^*}=\left(\frac{\lambda_t}{\lambda_{t-1}}\right)^{\frac{1}{\sigma-1}}\left[\frac{\tilde{P}_t^*}{\tilde{P}_{t-1}^*}\left(\frac{\tilde{S}_t^*}{\tilde{S}_{t-1}^*}\right)^{\frac{1}{\sigma-1}}\right]. \tag{21}$$

This price index is the discrete analog of equation (6). To obtain an estimate for the elasticity of substitution, we again combine this expression for the change in the cost of living from the demand system with an expression for the change in the cost of living from the unit expenditure function, and use the assumption of a constant aggregate utility function. When prices and expenditure shares are continuous, we can solve for this elasticity by using equation (8). With discrete changes in prices and expenditure shares, this problem is more complex, because we have two ways of computing the change in the cost of living from the unit expenditure function: the forward difference and the backward difference. The forward difference evaluates the *increase* in the price index from $t-1$ to t using the preferences of consumers in period $t-1$, and the backward difference uses the preferences of consumers period t to evaluate the *decrease* in the price index from t to $t-1$. These two approaches are identical for small changes, but as we move to consider discrete changes we need to realize that they will not yield the same answer in general.

In order to make this concrete, we first need to derive our forward and backward differences in price indexes. We can use equations (14) and (17) to write the forward change in the price index from period $t-1$ to period t in terms of the change in variety (λ_t/λ_{t-1}), the initial share of each common good in expenditure on all common goods (S_{kt-1}^*), and changes in prices (P_{kt}/P_{kt-1}) and demand ($\varphi_{kt}/\varphi_{kt-1}$) for all common goods:

$$\Phi_{t-1,t}^F=\left(\frac{\lambda_t}{\lambda_{t-1}}\right)^{\frac{1}{\sigma-1}}\frac{\mathbb{P}_t^*}{\mathbb{P}_{t-1}^*}=\left(\frac{\lambda_t}{\lambda_{t-1}}\right)^{\frac{1}{\sigma-1}}\left[\sum_{k\in\Omega_{t,t-1}}S_{kt-1}^*\left(\frac{P_{kt}/\varphi_{kt}}{P_{kt-1}/\varphi_{kt-1}}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}, \tag{22}$$

as shown in Section 7.1 of the appendix. Re-arranging (13) and using (14) and (17), we can write the backward change in the price index from period t to period $t-1$ as:

$$\Phi_{t,t-1}^B=\left(\frac{\lambda_{t-1}}{\lambda_t}\right)^{\frac{1}{\sigma-1}}\frac{\mathbb{P}_{t-1}^*}{\mathbb{P}_t^*}=\left(\frac{\lambda_{t-1}}{\lambda_t}\right)^{\frac{1}{\sigma-1}}\left[\sum_{k\in\Omega_{t,t-1}}S_{kt}^*\left(\frac{P_{kt-1}/\varphi_{kt-1}}{P_{kt}/\varphi_{kt}}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}, \tag{23}$$

where the algebra is again relegated to Section 7.1 of the appendix. The only variable not common to these two expressions is the expenditure share (S_{kt-1}^* versus S_{kt}^*). When evaluating the forward price change, we use the period $t-1$ expenditure shares, but we use the period t expenditure shares when evaluating the backward price change. Note that the terms in square brackets in (22)-(21) correspond to the changes in the CES price indexes for common goods.

Using the CES common good expenditure shares (17) and time reversibility ($\Phi^F_{t-1,t} = 1/\Phi^B_{t,t-1}$), the terms for the change in the price index for common goods in (22)-(21) can be written as:

$$\frac{\mathbb{P}^*_t}{\mathbb{P}^*_{t-1}} = \left[\Theta_{t-1,t} \sum_{k \in \Omega_{t,t-1}} S^*_{kt-1} \left(\frac{P_{kt}}{P_{kt-1}}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}, \tag{24}$$

$$\frac{\mathbb{P}^*_t}{\mathbb{P}^*_{t-1}} = \left[\Theta_{t,t-1} \sum_{k \in \Omega_{t,t-1}} S^*_{kt} \left(\frac{P_{kt}}{P_{kt-1}}\right)^{-(1-\sigma)}\right]^{-\frac{1}{1-\sigma}}, \tag{25}$$

$$\frac{\mathbb{P}^*_t}{\mathbb{P}^*_{t-1}} = \frac{\tilde{P}^*_t}{\tilde{P}^*_{t-1}} \left(\frac{\tilde{S}^*_t}{\tilde{S}^*_{t-1}}\right)^{\frac{1}{\sigma-1}}, \tag{26}$$

$$\Theta_{t-1,t} = \frac{1}{\sum_{k \in \Omega_{t,t-1}} S^*_{kt} \left(\frac{\varphi_{kt-1}}{\varphi_{kt}}\right)^{\sigma-1}}, \qquad \Theta_{t,t-1} = \frac{1}{\sum_{k \in \Omega_{t,t-1}} S^*_{kt-1} \left(\frac{\varphi_{kt}}{\varphi_{kt-1}}\right)^{\sigma-1}}, \tag{27}$$

as shown in Section 7.2 of the appendix.

The logs of the $\Theta_{t-1,t}$ and $\Theta_{t,t-1}$ are the discrete analogs to equation (9), which required that a share-weighted average of the log-differenced demand shocks to be zero in order to satisfy the assumption of constant aggregate utility. For discrete changes, we now have two such conditions corresponding to the backward and forward differences. In equation (27), in the left-hand term ($\Theta_{t-1,t}$), demand shocks starting in period t and going backwards in time to period $t-1$ (namely $(\varphi_{kt-1}/\varphi_{kt})^{\sigma-1}$) are weighted by the common goods expenditure shares in period t (S^*_{kt}). In the right-hand term ($\Theta_{t,t-1}$), demand shocks starting in period $t-1$ and going forward in time to period t (that is $(\varphi_{kt}/\varphi_{kt-1})^{\sigma-1}$) are weighted by the expenditure shares in period $t-1$ (S^*_{kt-1}). The assumption of constant aggregate utility implies that the product demand shocks cannot change the aggregate price indexes in equations (24)-(25):

$$\Theta^{1/(1-\sigma)}_{t-1,t} = \Theta^{-1/(1-\sigma)}_{t,t-1} = 1. \tag{28}$$

This assumption of constant aggregate utility has another intuitive interpretation. When this condition is satisfied, the consumer with time-varying demand for each product k has the same change in the aggregate price index in (24)-(26) as a consumer with time-invariant demand for each product k who faces the same prices and expenditure shares (since $\varphi_{kt} = \overline{\varphi}_k$ implies $\Theta^{1/(1-\sigma)}_{t-1,t} = \Theta^{-1/(1-\sigma)}_{t,t-1} = 1$ in equation (27)) .

Proposition 2. ***(Identification)*** *Given data for prices and expenditure shares $\{P_{kt}, S_{kt}\}$ for the sets of goods Ω_t, $\sigma \neq 1$, and variation in expenditure-share-weighted average price changes:*

$$\sum_{k \in \Omega_{t,t-1}} S^*_{kt-1} \ln\left(\frac{P_{kt}}{P_{kt-1}}\right) \neq \sum_{k \in \Omega_{t,t-1}} S^*_{kt} \ln\left(\frac{P_{kt}}{P_{kt-1}}\right) \neq \sum_{k \in \Omega_{t,t-1}} \frac{1}{N_{t,t-1}} \ln\left(\frac{P_{kt}}{P_{kt-1}}\right),$$

the assumption of constant aggregate CES preferences (equations (24)-(28)) identifies the elasticity of substitution ($\hat{\sigma}$) and the demand parameter for ($\hat{\varphi}_{kt}$) for each good k and time period t.

Proof. See Section 7.3 of the Appendix. □

The continuous-time case is simpler because equations (24) and (25) collapse a single equation (8), so one can solve for σ directly by combining this equation with equation (6), which is the continuous time version of equation (26). In discrete time, we now have two moment conditions to identify σ, so we are overidentified and therefore will use a generalized method of moments (GMM) estimator of the elasticity of substitution. Re-arranging (24)-(25), we obtain the following moment function for each time period:

$$m_t(\sigma) = \begin{pmatrix} m_t^1(\sigma) \\ m_t^2(\sigma) \end{pmatrix} = \begin{pmatrix} \ln\left[\sum_{k\in\Omega_{t,t-1}} S_{kt-1}^* \left(\frac{P_{kt}}{P_{kt-1}}\right)^{1-\sigma}\right] - (1-\sigma)\ln\left[\frac{\tilde{P}_t^*}{\tilde{P}_{t-1}^*}\left(\frac{\tilde{S}_t^*}{\tilde{S}_{t-1}^*}\right)^{\frac{1}{\sigma-1}}\right] \\ -\ln\left[\sum_{k\in\Omega_{t,t-1}} S_{kt}^* \left(\frac{P_{kt}}{P_{kt-1}}\right)^{-(1-\sigma)}\right] - (1-\sigma)\ln\left[\frac{\tilde{P}_t^*}{\tilde{P}_{t-1}^*}\left(\frac{\tilde{S}_t^*}{\tilde{S}_{t-1}^*}\right)^{\frac{1}{\sigma-1}}\right] \end{pmatrix} = \begin{pmatrix} -\ln(\Theta_{t-1,t}) \\ \ln(\Theta_{t,t-1}) \end{pmatrix}. \tag{29}$$

Taking expectations across time periods, we impose the moment condition:

$$M(\sigma) = \frac{1}{T}\sum_{t=1}^{T} m_t(\sigma) = 0. \tag{30}$$

We term the estimate of the elasticity of substitution that we obtain from this GMM procedure the "reverse-weighting" estimate ($\hat{\sigma}^{RW}$) because it minimizes the deviation between the forward and backward differences of the constant aggregate price index. The GMM estimator, $\hat{\sigma}^{RW}$, solves:

$$\hat{\sigma}^{RW} = \arg\min\left\{M\left(\sigma^{RW}\right)'\, \mathbb{W} M\left(\sigma^{RW}\right)\right\}, \tag{31}$$

where we weight the two moments for the forward and backward difference equally by using the identity matrix for the weighting matrix $\mathbb{W}$.

The reverse-weighting estimator has an intuitive economic interpretation. Each moment condition is a discrete version of equation (11) in the continuous-time case: one based on a forward difference and one based on a backward difference. If these two differences were identical, we would have $\Theta_{t-1,t}^{1/(1-\sigma)} = \Theta_{t,t-1}^{-1/(1-\sigma)} = 1$, and we could solve for σ as in the continuous case. However, since they will not in general be equal we minimize the deviations between these forward and backward differences and the change in the price index implied by the demand system $\Phi_{t-1,t}^{D}$.

Given any sample of data on prices and expenditure shares $\{P_{kt}, S_{kt}\}$, the reverse-weighting estimator identifies the value of the elasticity of substitution ($\hat{\sigma}^{RW}$) and the demand parameter ($\hat{\varphi}_{kt}^{RW}$) for each good k and time period t consistent with the assumption of constant aggregate CES preferences. However, to the extent that the demand parameters (φ_{kt}) are stochastic, there will be sampling error in any finite sample. Therefore we now provide conditions under which the reverse-weighting estimator $\{\hat{\sigma}^{RW}, \hat{\varphi}_{kt}^{RW}\}$ consistently estimates the true value of the elasticity of substitution (σ) and the demand parameter (φ_{kt}) for each good k and time period t. First, as demand shocks becomes small ($\varphi_{kt}/\varphi_{kt-1} \rightarrow 1$ for all k), the log of the aggregate demand shifters converges to zero ($\ln(\Theta_{t-1,t}) = \ln(\Theta_{t,t-1}) = 0$) in any finite sample. Using this consistent estimate for the elasticity of substitution ($\hat{\sigma}^{RW}$), we can recover consistent estimates of demand for each good k and time period t ($\hat{\varphi}_{kt}^{RW}$) from the expenditure share (3).

Proposition 3. ***(Consistency for Small Demand Shocks)*** *Suppose that we have data on prices and expenditure shares $\{P_{kt}, S_{kt}\}$ for the sets of goods Ω_t, $\sigma \neq 1$, and there exists variation in expenditure-share-weighted average* *price changes:*

$$\sum_{k\in\Omega_{t,t-1}} S^*_{kt-1}\ln\left(\frac{P_{kt}}{P_{kt-1}}\right) \neq \sum_{k\in\Omega_{t,t-1}} S^*_{kt}\ln\left(\frac{P_{kt}}{P_{kt-1}}\right) \neq \sum_{k\in\Omega_{t,t-1}} \frac{1}{N_{t,t-1}}\ln\left(\frac{P_{kt}}{P_{kt-1}}\right).$$

As demand shocks become small ($\varphi_{kt}/\varphi_{kt-1} \to 1$ for all k), the reverse-weighting estimator (31) consistently estimates the true elasticity of substitution ($\hat{\sigma}^{RW} \xrightarrow{p} \sigma$) and demand parameter for each good k and period t ($\hat{\varphi}^{RW}_{kt} \xrightarrow{p} \varphi_{kt}$).

Proof. See Section 4 of the Appendix. □

Second, as the number of common goods becomes large ($N_{t,t-1} \to \infty$) and for independently and identically distributed demand shocks ($\varphi_{kt}/\varphi_{kt-1} \sim$ i.i.d. $\left(1, \chi^2_\varphi\right)$), the log of the aggregate demand shifters converges in distribution to zero ($-\ln\left(\Theta_{t-1,t}\right) \xrightarrow{d} 0$ and $\ln\left(\Theta_{t,t-1}\right) \xrightarrow{d} 0$) for any finite number of time periods T. Using this consistent estimate for the elasticity of substitution ($\hat{\sigma}^{RW}$), we can again recover consistent estimates of demand for each good k and time period t ($\hat{\varphi}^{RW}_{kt}$) from the expenditure share (3).

Proposition 4. ***(Consistency for Large Number of Common Goods)*** *Suppose that we have data on prices and expenditure shares $\{P_{kt}, S_{kt}\}$ for the sets of goods Ω_t, $\sigma \neq 1$, and there exists variation in expenditure-share-weighted average price changes:*

$$\sum_{k\in\Omega_{t,t-1}} S^*_{kt-1}\ln\left(\frac{P_{kt}}{P_{kt-1}}\right) \neq \sum_{k\in\Omega_{t,t-1}} S^*_{kt}\ln\left(\frac{P_{kt}}{P_{kt-1}}\right) \neq \sum_{k\in\Omega_{t,t-1}} \frac{1}{N_{t,t-1}}\ln\left(\frac{P_{kt}}{P_{kt-1}}\right).$$

As the number of common goods becomes large ($N_{t,t-1} \to \infty$) and for independently and identically distributed demand shocks ($\varphi_{kt}/\varphi_{kt-1}$) $\sim$ i.i.d. $\left(1, \chi^2_\varphi\right)$, the reverse-weighting estimator (31) consistently estimates the elasticity of substitution ($\hat{\sigma}^{RW} \xrightarrow{p} \sigma$) and demand parameter for each good k and period t ($\hat{\varphi}^{RW}_{kt} \xrightarrow{p} \varphi_{kt}$).

Proof. See Section 7.5 of the Appendix. □

When demand shocks are small (Proposition 3) or the number of common goods is large and demand shocks are independently and identically distributed (Proposition 4), the shocks to demand for individual goods have no effect on the aggregate price index and both aggregate demand shifters converge in distribution to one ($\Theta_{t-1,t} \xrightarrow{d} 1$ and $\Theta_{t,t-1} \xrightarrow{d} 1$), which ensures that the reverse-weighting estimator consistently recovers the true parameter values. Having established these results, we are now in a position to define the unified price index:

Proposition 5. *The "unified price index" (UPI)—which is exact for the CES preference structure in the presence of changes in the set of goods, demand-shocks that do not affect aggregate utility, and discrete changes in prices and expenditure shares—is given by*

$$\Phi^U_{t-1,t} = \underbrace{\left(\frac{\lambda_t}{\lambda_{t-1}}\right)^{\frac{1}{\sigma^{RW}-1}}}_{\text{Variety Adjustment}} \underbrace{\left[\frac{\tilde{P}^*_t}{\tilde{P}^*_{t-1}}\left(\frac{\tilde{S}^*_t}{\tilde{S}^*_{t-1}}\right)^{\frac{1}{\sigma^{RW}-1}}\right]}_{\text{Common-Goods UPI}}. \tag{32}$$

Proof. Follows directly from equation (21) and Propositions 3 and 4. □

As in Feenstra (1994), our formula expresses the change in the cost of living as a function of variety-adjustment term and a common-goods price term, which we will term the CG-UPI. The former captures changes in the unit expenditure function due to product turnover, changes in the number of varieties, and new goods. The common-goods price index measures how changes in prices, demand-shifts, and product substitution affects a consumer's unit expenditure function. (λ_t), price, and share ratios as well as the elasticity of substitution and both indexes are time reversible for any value of σ. Critically, however, the UPI differs from that in Feenstra (1994), both in terms of the estimation of σ^{RW} and the formula for the common-goods price index, which differs from the formula derived in Sato (1976) and Vartia (1976). We explore the relation between the unified price index and existing price indexes in the next section.

3 Relation to Existing Price indexes

Obviously, the closest parallel to our index is Feenstra's (1994) CES exact price index, but it is also instructive to compare the unified price index with all major index numbers used by economists and statistical agencies, which we do in the following sections.

3.1 Exact CES Price Indexes

By far the most influential work in this area has been that of Feenstra (1994) who computed the exact price index given in equation (15) by using the Sato-Vartia weights to compute an exact price index corresponding to P_t^*/P_{t-1}^* and estimating σ for the variety correction term $((\lambda_t/\lambda_{t-1})^{1/(\sigma-1)})$ using a specification of the demand equation (4). However, Feenstra (1994) acknowledges that there is an inconsistency in this approach because the Sato-Vartia index used for $\mathbb{P}_t^*/\mathbb{P}_{t-1}^*$ imposes the restriction that demand is constant over time for each good ($\varphi_{kt} = \varphi_{kt-1} = \bar{\varphi}_k$ for all k and t), whereas the estimation of σ assumes that demand for goods changes over time ($\varphi_{kt} \neq \varphi_{kt-1}$ for some k and t).

To understand the problem, note that researchers can either assume constant or changing demand for goods. Under the assumption of constant demand for each good ($\varphi_{kt} = \varphi_{kt-1} = \bar{\varphi}_k$), we show in the proposition below that there is no need to estimate σ, because it can be recovered from observed prices and expenditure shares using the weights from the Sato-Vartia price index. Furthermore, the model is overidentified when demand is constant for each good, with the result that there exists an *infinite* number of approaches to measuring σ. If demand is indeed constant for each good ($\varphi_{kt} = \varphi_{kt-1} = \bar{\varphi}_k$), each of these approaches returns exactly the same value for σ. However, if demand for goods changes over time ($\varphi_{kt} \neq \varphi_{kt-1}$), and a researcher falsely assumes constant demand for each good, we show in the proposition below that each of these approaches returns a different value for σ. Even making the additional assumption that on average the change in demand for goods is zero does not eliminate the problem. Each of these approaches produces a different value for σ unless demand is constant for every good.

Proposition 6. ***(a)*** *Under the assumption that demand is constant for each good ($\varphi_{kt} = \varphi_{kt-1} = \bar{\varphi}_k$ for all k and t), the elasticity of substitution (σ) is uniquely identified from observed changes in prices and expenditure shares with no estimation. Furthermore, there exists a continuum of approaches to measuring σ, each of which weights prices and expenditure shares with different non-negative weights that sum to one, but returns the same value for σ.*

(b) *If demand for goods changes over time ($\varphi_{kt} \neq \varphi_{kt-1}$ for some k and t), but a researcher falsely assumes that demand for each good is constant, each of these alternative approaches returns a different value for σ, depending on which non-negative weights are used.*

Proof. See Section 7.6 of the Appendix. □

A corollary of Proposition 6 (b) is that if demand for each good is time varying, but a researcher falsely assumes that it is time invariant, the Sato-Vartia exact CES price index is not transitive: $\left(\mathbb{P}^*_t/\mathbb{P}^*_{t-1}\right)\left(\mathbb{P}^*_{t+1}/\mathbb{P}^*_t\right) \neq \left(\mathbb{P}^*_{t+1}/\mathbb{P}^*_{t-1}\right)$. The reason is that the implied σ for the longer difference between periods $t-1$ and $t+1$ is not the same as the two implied σ's for the two shorter differences between periods $t-1$ and t and periods t and $t+1$. Indeed, this proposition provides a simple explanation for why the exact CES price index is not transitive in practice and undermines welfare calculations because the failure of transitivity means that the Sato-Vartia elasticity in one period given by equation (104) differs from that in another period. Therefore, the utility function cannot be constant. By contrast, our common-goods price index given in equation (32) is transitive if the set of goods is constant, even if demand for each good is time-varying.

Under the alternative assumption that demand for each good changes over time ($\varphi_{kt} \neq \varphi_{kt-1}$ for some k and t), the Sato-Vartia price index for $\mathbb{P}^*_t/\mathbb{P}^*_{t-1}$ is invalid, because its assumption of constant demand for each good is not satisfied. In the proposition below, we now provide the generalization of the Sato-Vartia price index to take into account changes in demand for each good:

Proposition 7. *Assuming that demand for goods changes over time ($\varphi_{kt} \neq \varphi_{kt-1}$ for some k and t) and given consistent estimates of the elasticity of substitution ($\hat{\sigma}^{RW}$) and demand for each good ($\hat{\varphi}^{RW}_{kt}$) from the reverse-weighting estimator, the exact common-goods CES price index can be expressed as:*

$$\frac{\mathbb{P}^*_t}{\mathbb{P}^*_{t-1}} = \prod_{k \in \Omega_{t,t-1}} \left(\frac{P_{kt}/\hat{\varphi}^{RW}_{kt}}{P_{kt-1}/\hat{\varphi}^{RW}_{kt-1}} \right)^{\omega^*_{kt}}, \tag{33}$$

$$\hat{\varphi}^{RW}_{kt} = \frac{P_{kt}}{\tilde{P}^*_t} \left(\frac{S_{kt}}{\tilde{S}^*_t} \right)^{\frac{1}{\hat{\sigma}^{RW}-1}}, \qquad \omega^*_{kt} = \frac{\frac{S^*_{kt}-S^*_{kt-1}}{\ln S^*_{kt} - \ln S^*_{kt-1}}}{\sum_{\ell \in \Omega_{t,t-1}} \frac{S^*_{\ell t}-S^*_{\ell t-1}}{\ln S^*_{\ell t} - \ln S^*_{\ell t-1}}}, \qquad \sum_{k \in \Omega_{t,t-1}} \omega^*_{kt} = 1. \tag{34}$$

Proof. See the appendix. □

The Sato-Vartia formula is a special case of the exact common-goods CES price index (33) in which $\varphi_{kt}/\varphi_{kt-1} = 1$ for all k and t. In this special case, the change in the common goods price index can be expressed solely in terms of observed prices (P_{kt}) and expenditure share weights (ω^*_{kt}). In the more general formula (33), the common goods price index also depends on the reverse-weighting estimates of product demand ($\hat{\varphi}^{RW}_{kt}$) and hence the elasticity of substitution ($\hat{\sigma}^{RW}$). We showed above how these reverse-weighting estimates are determined from observed price and expenditure share data.[2]

The exact common goods CES price index incorporating changes in product demand (33) highlights a hitherto neglected source of bias in the measurement of changes in the cost of living. Taking logarithms in

[2]As considered in Banerjee (1983) for the special case of constant demand for each good ($\varphi_{kt} = \bar{\varphi}_k$), the Sato-Vartia weights (ω^*_{kt}) are only one of a broader class of weights that can be used to construct the exact common-goods CES price index.

(33), the change in the common goods CES price index can be written as the change in the conventional Sato-Vartia price index and a bias term:

$$\ln \Phi_t = \ln \Phi_t^{SV} - \left[\sum_{k \in \Omega_{t,t-1}} \omega_{kt}^* \ln \left(\frac{\hat{\varphi}_{kt}^{RW}}{\hat{\varphi}_{kt-1}^{RW}} \right) \right]. \tag{35}$$

Proposition 8. *The Sato-Vartia CES Price Index will overstate inflation relative to the true common-goods CES price index if demand shocks ($\ln\left(\hat{\varphi}_{kt}^{RW} / \hat{\varphi}_{kt-1}^{RW}\right)$) are positively correlated with the Sato-Vartia expenditure share weights (ω_{kt}^*), and understate it if they are negatively correlated. The Sato-Vartia index is exact only if demand shocks ($\ln\left(\hat{\varphi}_{kt}^{RW} / \hat{\varphi}_{kt-1}^{RW}\right)$) are orthogonal to the Sato-Vartia expenditure share weights (ω_{kt}^*).*

Proof. See Section 7.8 of the Appendix. □

In general, demand shocks ($\ln\left(\hat{\varphi}_{kt}^{RW} / \hat{\varphi}_{kt-1}^{RW}\right)$) will not be orthogonal to the Sato-Vartia expenditure share weights (ω_{kt}^*), because the Sato-Vartia weights (34) depend on changes in expenditure shares ($\ln\left(S_{kt}^* / S_{kt-1}^*\right)$), and these changes in expenditure shares are themselves functions of demand shocks ($\ln\left(\hat{\varphi}_{kt}^{RW} / \hat{\varphi}_{kt-1}^{RW}\right)$) as well as price shocks ($\ln\left(P_{kt} / P_{kt-1}\right)$). Therefore the Sato-Vartia expenditure share weights (ω_{kt}^*) are in general correlated with demand shocks ($\ln\left(\hat{\varphi}_{kt}^{RW} / \hat{\varphi}_{kt-1}^{RW}\right)$). The sign and magnitude of this correlation—and hence the sign and magnitude of the bias in the Sato-Vartia CES Price Index—depends on the variance of demand shocks relative to the variance of price shocks and the covariance between these two shocks.

Another way of thinking about this new source of bias in the measurement of changes in the cost of living is as follows. If the CES demand equation requires changes in demand for each good to explain observed changes in expenditure shares given observed changes in prices, the consumer's relative valuation of goods is changing over time. In this case, the CES price index cannot simply be computed as an expenditure-share weighted-average of the observed changes in prices because this does not take into account the changing relative valuation of goods. Instead, the observed changes in prices must be adjusted by these changing relative valuations in order to compute the overall change in the CES price index as in equation (33). Intuitively, duality implies that *both* the CES demand equation and price index are derived from a common underlying utility function. Therefore, allowing for changes in demand for each good in one of these equations (demand) and omitting changes in demand in the other (the price index) is inconsistent with duality.

In conclusion, Propositions 6 and 7 imply that there are two major differences between our index (32) and the Feenstra exact CES index that is commonly used. First, there is an inconsistency in the Feenstra exact CES index, because the Sato-Vartia index used for the common goods price index ($\mathbb{P}_t^* / \mathbb{P}_{t-1}^*$) imposes the restriction that demand for each good is constant over time, whereas the estimation of σ for the variety correction term ($(\lambda_t / \lambda_{t-1})^{1/(\sigma-1)}$) assumes that demand for each good changes over time. Second, if demand for each good is in fact time-varying, but a researcher falsely assumes that demand for each good is constant, the choice of the Sato-Vartia weights for the common goods price index ($\mathbb{P}_t^* / \mathbb{P}_t^*$) itself arbitrarily yields one of an infinite number of possible values or σ (from Proposition 6), which in general will differ from the estimate of σ from the demand equation used for the variety correction term ($(\lambda_t / \lambda_{t-1})^{1/(\sigma-1)}$).

Finally, even if one decided that the Sato-Vartia elasticity based on price and share changes from two time periods was correct, there is no guarantee that one would obtain the same elasticity if one applied the same procedure using data drawn from different time periods. This "multi-elasticity" structure both within a time period and across time periods is not consistent with the CES utility function.

3.2 Conventional Price Indexes

In relating economic indexes to those actually used, the first question we need to ask is how do we measure prices? The International Labor Organization (ILO) did a survey of 68 countries around the world and found that the method proposed by Dutot (1738) is still the most prominent one for measuring price changes (Stoevska (2008)).[3] This index is the ratio of a simple average of prices in two periods:

$$\Phi_{t-1,t}^{D} \equiv \frac{\frac{1}{N_{t,t-1}} \sum_{k\in\Omega_{t,t-1}} P_{kt}}{\frac{1}{N_{t,t-1}} \sum_{k\in\Omega_{t,t-1}} P_{kt-1}} = \sum_{k\in\Omega_{t,t-1}} \frac{P_{k,t-1}}{\sum_{k\in\Omega_{t,t-1}} P_{kt-1}} \left(\frac{P_{kt}}{P_{k,t-1}}\right) \tag{36}$$

As the above formula shows, this index is simply a price-weighted average change in prices, which makes little sense in terms of economic theory.

A price-weighted average of price changes is a sufficiently problematic way of measuring inflation that most statistical agencies do not just compute unweighted averages of prices in two periods, but select their sample of price quotes based on the largest selling products in the first period. If we think that the probability that a statistical agency picks a product for inclusion in its sample of prices is based on its purchase frequency ($C_{\ell,t-1} / \sum_{k\in\Omega_{t-1,t}} C_{k,t-1}$), then the Dutot index, as it is typically implemented, becomes the more familiar Laspeyres:

$$\Phi_{t-1,t}^{L} \equiv \frac{\sum_{k\in\Omega_{t,t-1}} C_{k,t-1} P_{kt}}{\sum_{k\in\Omega_{t,t-1}} C_{k,t-1} P_{kt-1}} = \sum_{k\in\Omega_{t,t-1}} \frac{C_{k,t-1} P_{k,t-1}}{\sum_{k\in\Omega_{t,t-1}} C_{k,t-1} P_{kt-1}} \left(\frac{P_{kt}}{P_{k,t-1}}\right) = \sum_{k\in\Omega_{t,t-1}} S_{kt-1}^{*} \frac{P_{kt}}{P_{kt-1}}. \tag{37}$$

Written this way, it is clear that Laspeyres index is a special case of our CES price index (22) in which the utility gain of new goods is exactly offset by the losses from disappearing goods ($\lambda_t / \lambda_{t-1} = 1$), the elasticity of substitution is equal to zero and demand for each good is constant ($\varphi_{kt} / \varphi_{kt-1} = 1$).

The Carli index, used by 19 percent of countries, is another popular index that can be thought of as a variant of the Laspeyres. The formula for the Carli index is

$$\Phi_{t-1,t}^{C} \equiv \sum_{k\in\Omega_{t,t-1}} \frac{1}{N_{t,t-1}} \left(\frac{P_{kt}}{P_{k,t-1}}\right) \tag{38}$$

This index is identical to the Laspeyres if all goods have equal expenditure shares. However, as with the Dutot, it is important to remember that statistical agencies are more likely to select a good for inclusion in the sample with a past high sales share ($S_{k,t-1}^{*}$) for inclusion, this index again collapses back to the Laspeyres formula.

[3] 41 percent of countries use this index although historically its popularity was much higher. For example, all U.S. inflation data prior to 1999 is based on this index, and Belgian, German, and Japanese data continues to be based on it. The odd formula may help explain why Dutot (1738) found it difficult to identify the impact of the money supply on the French price level: a problem still vexing macroeconomists today. The ILO report can be accessed here: http://www.ilo.org/public/english/bureau/stat/download/cpi/survey.pdf

Similarly, the Paasche index is closely related to the Laspeyres with the only difference that it weights price changes from $t-1$ to t by their expenditure shares in the *end* period t:

$$\Phi^{P}_{t-1,t} = \frac{\sum_{k\in\Omega_{t,t-1}} P_{kt}C_{kt}}{\sum_{k\in\Omega_{t,t-1}} P_{kt-1}C_{kt}} = \left[\sum_{k\in\Omega_{t,t-1}} S^{*}_{kt}\left(\frac{P_{kt}}{P_{kt-1}}\right)^{-1}\right]^{-1}. \tag{39}$$

We can also think of the Paasche index as is a special case of the CES price index (23) in which we apply the same parameter restrictions to derive the Laspeyres index.[4]

Finally, the Jevons index, which forms the basis of the lower level of the U.S. Consumer Price Index, is the second-most popular index currently in use, with 37 percent of countries building their measures of price changes based on it.[5] The index is constructed by taking an unweighted geometric mean of price changes from $t-1$ to t:

$$\Phi^{J}_{t-1,t} = \prod_{k\in\Omega_{t,t-1}}\left(\frac{P_{kt}}{P_{kt-1}}\right)^{\frac{1}{N_{t,t-1}}} = \frac{\tilde{P}^{*}_{t}}{\tilde{P}^{*}_{t-1}}. \tag{40}$$

This formula is a special case of the change in the CES price index (21) if we assume the utility gain of new goods is exactly offset by the losses from disappearing goods ($\lambda_t/\lambda_{t-1} = 1$), demand for each good is constant ($\varphi_{kt}/\varphi_{kt-1} = 1$), all goods have the same expenditure share (equal to $1/N_{t,t-1}$), and then we take the limit as the elasticity of substitution approaches one. Once again, statistical agencies typically choose products based on their historic sales shares. In this case the Jevons

$$\Phi^{CD}_{t-1,t} = \prod_{k\in\Omega_{t,t-1}}\left(\frac{P_{kt}}{P_{kt-1}}\right)^{S^{*}_{k,t-1}}, \tag{41}$$

which Konyus (Konüs) and Byushgens (1926) proved was exact for the Cobb-Douglas (1928) functional form. This price index which is a special case of the CES price index when the elasticity of substitution equals one, demand for each good is constant, and there are changes in variety.

Existing measures of prices are therefore special cases of the unified approach developed in this paper, and biases can be thought of in terms of parameter restrictions. For example, "substitution bias" arises from building a price index using the wrong elasticity of substitution (σ). Most studies of consumer behavior suggest that this elasticity is greater than one, but in Laspeyres and Paasche indexes it arrises because the elasticity is assumed to be zero. The recent move to the Jevons index by many countries reduced the bias by switching to an elasticity of one, but it probably did not eliminate it if one believes the elasticity to be larger than one. Our index corrects for this shortcoming in previous indexes by letting the data determine the correct elasticity.

"Variety" or "New Goods Bias" arrises from the assumption that $\lambda_t/\lambda_{t-1} = 1$. While this is typically stated in terms of these indexes being biased because these three indexes do not allow for new goods, we can see theoretically that this is not technically correct. The absence of new goods would correspond to $\lambda_t = 1$. While it is true that if there are no new or exiting goods, we will have $\lambda_t = \lambda_{t-1} = 1$, the validity of Laspeyres, Paasche, and Jevons indexes depends on a slightly weaker assumption: $\lambda_t/\lambda_{t-1} = 1$, which means that which the utility gain of new goods is exactly offset by the losses from disappearing goods. The

[4] To derive (39) from (23), we use $\Phi_{t-1,t} = 1/\Phi_{t,t-1}$, set $\varphi_{kt}/\varphi_{kt-1} = 1$ for all k, and set $\sigma = 0$.

[5] The percentages do not sum to 100 because 3 percent of sample respondents used other formulas.

fact that we tend to think that price per unit quality of new goods exceed that of disappearing goods—one gets more utility from paying $1,000 for a computer today than ten years ago—implies that this assumption is wrong since that would mean $\lambda_t/\lambda_{t-1} < 1$. Since our index incorporates new and disappearing goods into the price index.

The third source of bias is novel and arises from the assumption that tastes for each good are time invariant ($\varphi_{kt}/\varphi_{kt-1} = 1$). It is well known that a Laspeyres index tends to overstate inflation because it does not allow for product substitution. Mechanically, this arrises in formula (37) whenever prices fluctuate from high and low levels because lagged expenditure shares will be high when prices are low and low when prices are high. Thus, the Laspeyres index will overweight the price increases even if prices just fluctuate around some mean value, resulting in an upward bias. Interestingly, the extent of this bias is dependent on how much price changes rise with demand shifts. To see this, assume that period $t-1$ demand shifts are fixed and imagine that there are a series of demand shocks in period t. Simple inspection of equation (37) reveals that holding fixed the price changes the value of the Laspeyres index will not be affected by the magnitude of the demand shocks since they do not enter into the formula. However, they do enter into equation (22) by offsetting any price increase. Indeed, if the demand shocks were of equal magnitude as the price increases, the demand-adjusted CES index would register no price increase. Thus, the true CES index is falling in the co-movement between price and demand shocks because a price increase reduces utility less if it occurs when consumers are obtaining more utility per unit of the good. Since the Laspeyres index is unaffected by this co-movement but the true CES is falling in it, the difference between the two (the bias) must be rising in it.[6]

3.3 Superlative Indexes

Interestingly, the two remaining "superlative" price indexes are also closely related to the CES. The commonly used Fisher and Törnqvist are simply quadratic means of order r of the Laspeyres and Paasche indexes with the only difference between the choice of r (Diewert 1976). Thus, one can think of them as different ways of aggregating the same information contained equations (22) and (23) under the assumption of no changes in tastes for each good and no changes in the set of available goods. For example, the Fisher Index is the geometric mean of the Laspeyres and Paasche indexes,

$$\Phi^F_{t-1,t} = \left(\Phi^L_{t-1,t}\Phi^P_{t-1,t}\right)^{1/2}, \tag{42}$$

which is nothing more than a quadratic mean of order 2 of the two expressions for the change in the CES price index in equations (22) and (23) for the special case in which all goods are common ($\lambda_t = \lambda_{t-1} = 1$), the elasticity of substitution is zero, and demand for each good is constant ($\varphi_{kt}/\varphi_{k,t-1} = 1$). If we had relaxed the last assumption, the Fisher index could be written as a geometric average of equations (22) and (23).

Closely related to the Fisher index is the Törnqvist index. In Appendix 7.9, we re-derive the formula for the Törnqvist under the assumption that consumers have a translog expenditure function and demand

[6]The same argument applies in reverse for the Paasche index, whose bias is increasing in the co-movement of demand and price shocks.

is time varying. In this case, we can write the Törnqvist index

$$\Phi^{T}_{t-1,t} = \prod_{k \in \Omega_{t,t-1}} \left(\frac{P_{kt} / \varphi_{kt}}{P_{kt-1} / \varphi_{k,t-1}} \right)^{\frac{1}{2}\left(S^{*}_{kt-1} + S^{*}_{kt}\right)} . \tag{43}$$

Again, the standard Törnqvist index is just a form of the generalized Törnqvist in the special case in which there are no demand shocks ($\varphi_{kt} / \varphi_{k,t-1} = 1$). Another way of looking at the Törnqvist index is to realize that it is just a geometric average of Cobb-Douglas price indexes defined in equation (41) evaluated at times $t-1$ and t. Therefore, the Törnqvist index corresponds to a special case of only common goods ($\lambda_t = \lambda_{t-1} = 1$), constant demand for each good ($\varphi_{kt} / \varphi_{k,t-1} = 1$), and an elasticity of substitution equal to one. In other words, superlative indexes are simply different averages of the same underlying building blocks of the unified price index under the common assumptions of no change in the set of goods and no demand shocks.[7]

4 Heterogeneous Consumers

There are a number of objections that have been raised to using the CES setup. The easiest to dismiss is the arising from the fact that if consumers had CES preferences they would demand all goods, while in reality we observe consumers that typically have a preferred variety as has been developed in the discrete choice literature following McFadden (1974). This objection is not really substantive as Anderson, de Palma, and Thisse (1992) showed that the CES preferences of the representative consumer is identical to the aggregate behavior of all consumers in a random utility model in which heterogeneous consumers only demand their preferred good.

A second potential objection is that CES imposes strong assumptions in the form of symmetric substitution elasticities, homotheticity, and the independence of irrelevant alternatives (IIRA). This IIRA property implies that the relative sales of any two varieties depends only on their relative characteristics and not on the characteristics of other varieties supplied to the market. Relaxing these assumptions was one of the key motivations for the random coefficients model with a continuum of unobserved types in Berry, Levinsohn and Pakes (1995). In this section, we also extend the random utility model to allow for multiple types of consumers with different substitution and preference parameters. This extension relaxes the assumptions of symmetry, homotheticity and IIRA using a discrete number of types as in the mixed logit model of McFadden and Train (2000).

In particular, we partition into different types indexed by $r \in \{1, \dots, R\}$. The utility of an individual i of type r who consumes C^{r}_{iu} units of product u is:

$$\mathbb{U}^{r}_{i} = z^{r}_{ik} \varphi^{r}_{k} C^{r}_{ik}, \tag{44}$$

where φ^{r}_{k} captures type-r consumers' common tastes for product k; z^{r}_{ik} captures idiosyncratic consumer tastes for each product; and we have omitted the time subscript t on each variable to simplify notation.

[7] Another way of of making the same point is that the square root of (22) and (23) in the special case of only common goods ($\lambda_t = \lambda_{t-1} = 1$) and constant product appeal ($\varphi_{kt} = \varphi_{kt-1} = 1$ for all k) yields a quadratic mean of order $2(1-\sigma)$ price index. In the special case of $\sigma = 0$ this corresponds to a Fisher index, while in the special case of $\sigma = 1$ this corresponds to a Törnqvist index.

Since the consumer only consumes their preferred good, their budget constraint implies:

$$C_{ik}^{r} = \frac{E_{i}^{r}}{P_{k}^{r}}, \tag{45}$$

where E_i^r is the consumer's expenditure and P_k^r is the price of the good available to the consumer of type r. Using this result, utility (44) can be re-written in the indirect form as:

$$\mathbb{U}_{i}^{r} = z_{ik}^{r} \left(\varphi_{k}^{r} / P_{k}^{r} \right) E_{i}^{r}. \tag{46}$$

These idiosyncratic tastes are assumed to have a Fréchet (Type-II Extreme Value) distribution:

$$G(z) = e^{z^{-\theta^{r}}}, \tag{47}$$

where we allow the shape parameter determining the dispersion of idiosyncratic tastes (θ^r) to vary across types. We normalize the scale parameter of the Fréchet distribution to one, because it affects consumer expenditure shares isomorphically to the consumer tastes parameter φ_k^r. Using the monotonic relationship between idiosyncratic tastes and utility, we have:

$$z_{ik}^{r} = \frac{\mathbb{U}_{i}^{r}}{\left(\varphi_{k}^{r} / P_{k}^{r} \right) E_{i}^{r}}.$$

Therefore, the distribution of utility from product k for individual i is:

$$G_{ik}^{r}\left(\mathbb{U}^{r}\right) = \exp\left(\left(\frac{\mathbb{U}_{i}^{r} P_{k}^{r}}{\varphi_{k}^{r} E_{i}^{r}} \right)^{-\theta^{r}} \right). \tag{48}$$

From this distribution of utility (48), the probability that an individual i of type r chooses product u is the same across all individuals of that type and equal to:

$$S_{ik}^{r} = S_{k}^{r} = \frac{\left(P_{k}^{r} / \varphi_{k}^{r} \right)^{-\theta^{r}}}{\sum_{k=1}^{N} \left(P_{k}^{r} / \varphi_{k}^{r} \right)^{-\theta^{r}}}, \tag{49}$$

The expected utility of consumer i of type r is:

$$\mathbb{E}\left[\mathbb{U}^{r}\right] = \gamma^{r} \left[\sum_{k=1}^{N} \left(E_{i}^{r} \right)^{\theta^{r}} \left(P_{k}^{r} / \varphi_{k}^{r} \right)^{-\theta^{r}} \right]^{\frac{1}{\theta^{r}}}, \qquad \gamma^{r} = \Gamma\left(\frac{\theta^{r} - 1}{\theta^{r}} \right), \tag{50}$$

where $\Gamma(\cdot)$ is the Gamma function. This expected utility can be re-written as:

$$\mathbb{E}\left[\mathbb{U}^{r}\right] = \frac{E_{i}^{r}}{\mathbb{P}^{r}}, \tag{51}$$

where $\mathbb{P}^r$ is the price index for consumers of type r:

$$\mathbb{P}^{r} = \left(\gamma^{r}\right)^{-1} \left[\sum_{k=1}^{N} \left(P_{k}^{r} / \varphi_{k}^{r} \right)^{-\theta^{r}} \right]^{-\frac{1}{\theta^{r}}}. \tag{52}$$

Total expenditure on a product k across all consumers i of type r is:

$$E_{u}^{r} = \sum_{i} E_{ik}^{r} = \sum_{i} S_{k}^{r} E_{i}^{r} = S_{k}^{r} E^{r},$$

which can be re-written as:

$$E_{k}^{r} = \left(P_{k}^{r} / \varphi_{k}^{r} \right)^{-\theta^{r}} \left(\mathbb{P}^{r} \right)^{\theta^{r}} E^{r}, \tag{53}$$

where $\mathbb{P}^r$ is again the price index (52) for consumer of type r.

The key point to realize is that if we change notation and define $\theta^r = \sigma^r - 1$ and assume that there is only one type (r) of consumers, equations (49) and (52) become identical to demand and price index equations we derived in the CES case. Thus, the CES demand system and its "love-of-variety" property can be thought of as a means of aggregating "ideal-type" consumers who only consume one of each type of variety.

Proposition 9. *Given data on prices and on expenditure of consumers of each type r, the mixed random utility model defined by the indirect utility function (44) and Type-II Extreme Value distributed idiosyncratic tastes (47) with shape parameter θ^r is isomorphic to a constant elasticity of substitution (CES) model in which consumers of different types have different demand parameters (φ_u^r) and elasticities of substitution (σ^r). This mixed random utility model implies an expenditure function (53) and price index (52) for consumers of a given type r that are isomorphic to those in a mixed CES model with multiple consumer types, where $\theta^r = \sigma^r - 1$.*

Proof. The proposition follows immediately from the expenditure function (53) and price index (52), substituting $\theta^r = \sigma^r - 1$.

□

Given data on prices (P_k^r) and expenditures (E_k^r) for consumers of different observable types r (*e.g.*, consumers from different regions or income quantiles), our CES-based methodology can be used to estimate the elasticity of substitution ($\sigma^r = 1 + \theta^r$) and product appeal (φ_k^r) for each type. The share of expenditure on product u across consumers of all types is:

$$S_k = \sum_{r \in R} S^r S_k^r, \tag{54}$$

where S^r is the share of consumers of type r in total expenditure.

This model with multiple types of consumers generates much richer predictions for cross-price elasticities than the CES model with a single consumer type. Summing consumer demands across types using (53) and using the definition of the product's overall expenditure share (54), the cross-price elasticity of the demand for product k with respect to the price for product k' is given by:

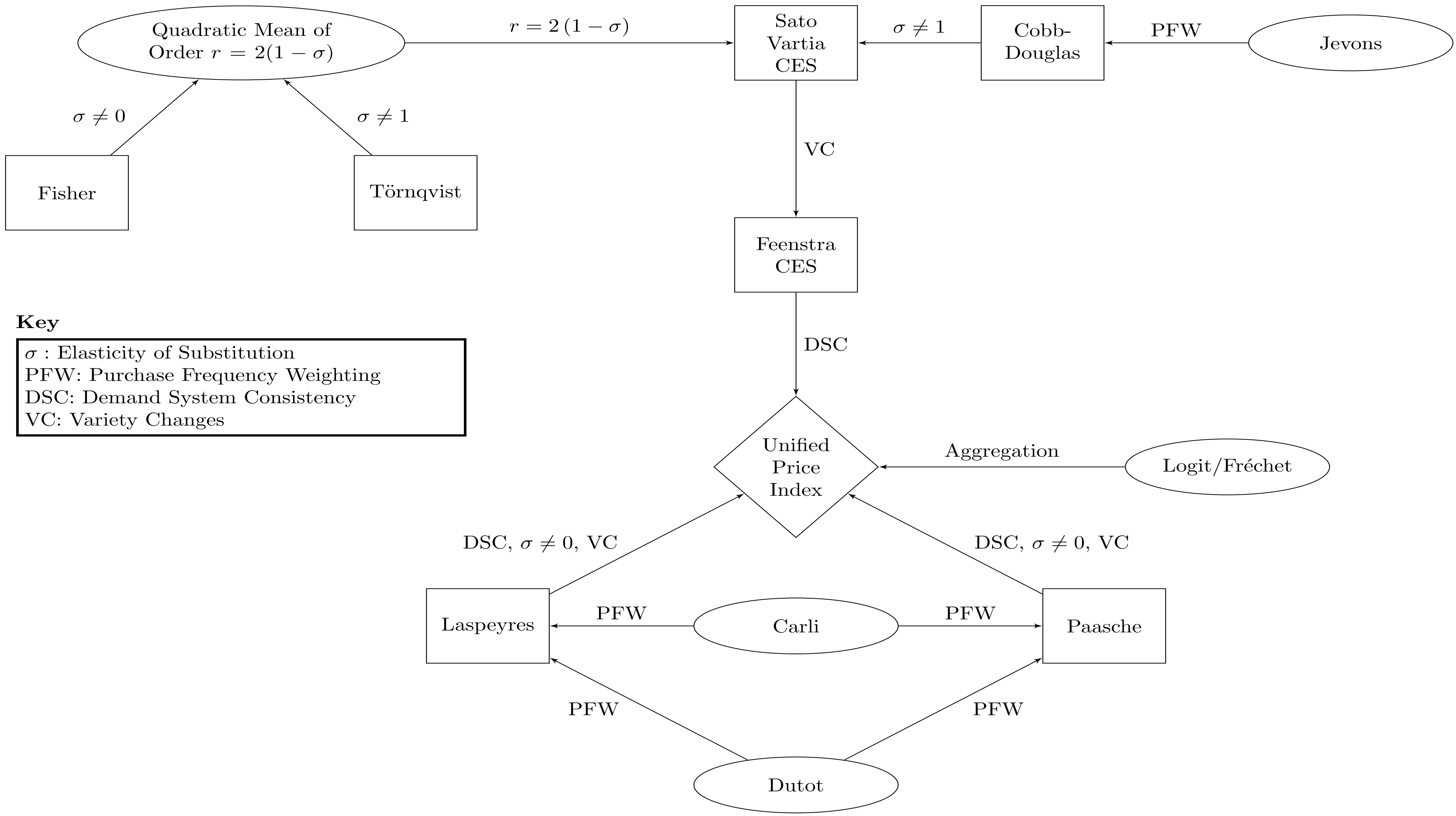
$$\frac{\partial C_k}{\partial P_{k'}} \frac{P_{k'}}{C_k} = \sum_r S^r \theta^r \frac{S_k^r S_{k'}^r}{S_k}. \tag{55}$$

Considering data on multiple markets with different shares of each consumer type, these cross-price elasticities will vary across markets depending on the shares of each consumer type in overall expenditure and the share of each product in total expenditure by each consumer type. Furthermore, the IIRA property will no longer necessarily hold across these different markets. The relative sales of two products across markets will depend not only on the characteristics of those products, but also on the shares of each consumer type in overall expenditure and the share of each product in total expenditure by each consumer type (which depends on the characteristics of other products). Finally, partitioning consumer types by income, the variation in the substitution and preference parameters across types allows for non-homotheticities in preferences across consumer types.

5 Results

Figure 1 summarizes how all major indexes and are related to the unified index. Most indexes (such as the Dutot, Carli, Laspeyres, Paasche, Jevons, Cobb-Douglas, Sato-Vartia-CES, Feenstra-CES) are simply special cases of our index. One can think of the "statistical approach" to index numbers, therefore, as versions of the unified approach in which statisticians make different parameter restrictions, ignore certain parts of the data (e.g., new goods), ignore certain implications of the model, and fail to sample based on purchase frequencies. Exact CES price indexes are based on no demand shocks, and superlative indexes are simply different weighted averages of the same building blocks as those of the unified index under the assumption of no change in the set of goods or consumer tastes. The relaxation of all of these assumptions and restrictions results in the unified approach.

Figure 1: Relation Between Existing Indexes and the UPI



Our next task is to show how the unified price index performs. We do this in two ways. First, we conduct a Monte Carlo simulation that enables us to examine how well the reverse-weighting estimator performs on simulated data. This exercise enables us to abstract from data problems and see how well our estimation methods work when we know the true parameters. Second, we turn to actual bar-code data to test the properties of the index on actual data.

5.1 Monte Carlo Simulation

Our data generating process arises from a model economy with CES demand and a standard supply-side in the form of monopolistic competition and constant marginal costs. We first assume true values of the model's parameters (the elasticity of substitution σ) and its structural residuals (demand for each good and marginal cost). We next solve for equilibrium prices and expenditure shares in this economy. Finally, we assume that a researcher only observes data on these prices and expenditure shares and uses our reverse-weighting estimator to estimate the elasticity of substitution and demand for each good. We compare our parameter estimates with the model's true parameters. We also compare the estimated change in the cost of living using our unified approach with the true change in the model and with the estimated change based on the standard price indexes used by statistical agencies and economists.

We consider an economy with goods $k \in \Omega_t$ and periods $t \in T$. A subset of the goods in each period $\Omega^C \subseteq \Omega_t$ is common to all periods $t \in T$. The remainder of the goods in period t ($\Omega_t^X \subset \Omega_t$) are available in period t but not in any other period, such that $\Omega_t = \left\{\Omega^C \cup \Omega_t^X\right\}$. Therefore the universe of goods is the union of the set of common goods and the sets of new goods supplied each period: $\Omega = \left\{\Omega^X \cup \Omega_1^X \cup \Omega_2^X \ldots \cup \Omega_T^X\right\}$. We assume 50 time periods (N^T) and let the number of common goods (N^C) range from 10 to 500. We assume that the number of new goods each period (N^X) is 20 percent of the number of common goods, which is below the entry and exit rates typically observed in bar-code data.

We assume the following values for the model's parameters. We set the elasticity of substitution equal to 4, which is consistent with estimates using U.S. data in Bernard, Eaton, Jensen and Kortum (2004). The time-varying demand shifters (φ_{kt}) are drawn for each good and time period from an independent log normal distribution: $\ln \varphi_{kt} \sim \mathcal{N}\left(0, \chi_\varphi\right)$. We also draw time-varying marginal cost (b_{kt}) for each good and time period from an independent log normal distribution: $\ln b_{kt} \sim \mathcal{N}\left(0, \chi_b\right)$. In this specification, both demand shocks ($\varphi_{kt}/\varphi_{kt-1}$) and supply shocks ($b_{kt}/b_{kt-1}$) are independently and log normally distributed.[8]

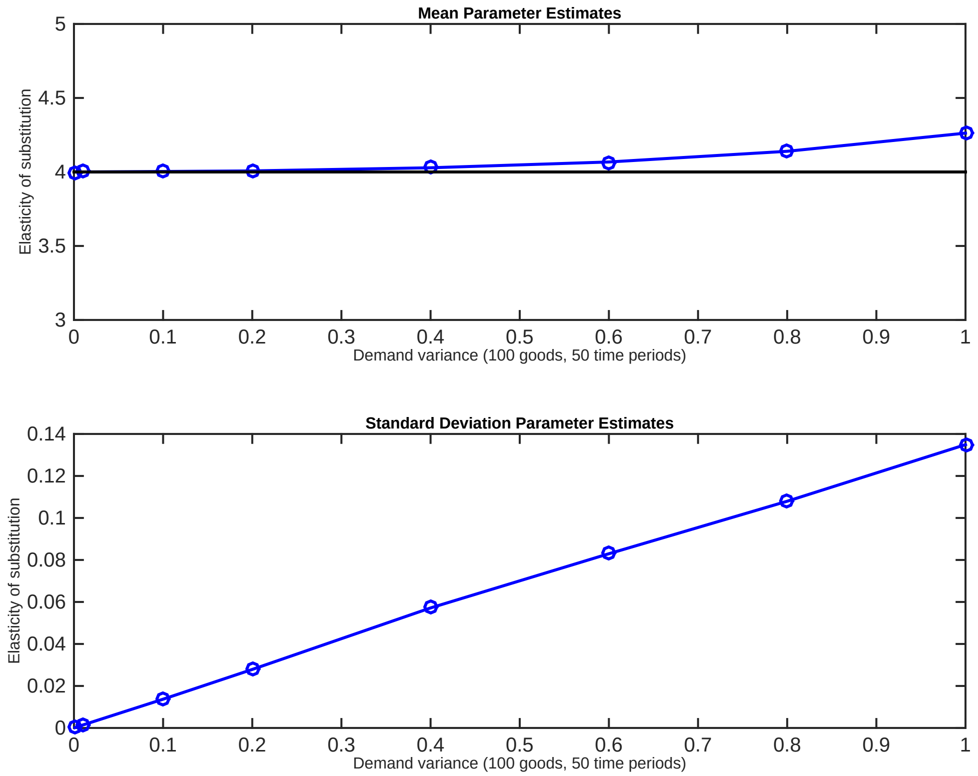
For a given number of time periods and goods (N^T, N^C, N^X), we undertake 250 replications of the model. In each replication, we first draw random realizations for product appeal (φ_{kt}) and marginal cost (b_{kt}) and solve for unique equilibrium prices, expenditure shares and common goods expenditure shares (P_{kt}, S_{kt}, S_{kt}^*) from the following system of equations:

$$P_{kt} = \frac{\sigma}{\sigma - 1} b_{kt},$$

$$S_{kt} = \frac{\left(P_{kt}/\varphi_{kt}\right)^{1-\sigma}}{\sum_{\ell \in \Omega_t} \left(P_{\ell t}/\varphi_{\ell t}\right)^{1-\sigma}} \qquad S_{kt}^* = \frac{\left(P_{kt}/\varphi_{kt}\right)^{1-\sigma}}{\sum_{\ell \in \Omega_{t,t-1}} \left(P_{\ell t}/\varphi_{\ell t}\right)^{1-\sigma}}.$$

[8] We assume that firms find it profitable to supply all goods Ω_t in each time period t, which implicitly corresponds to an assumption that any fixed costs of production are sufficiently small that the variable profits from supplying all goods are positive.

Figure 2: Mean and Standard Deviation of Estimated Elasticity of Substitution for Different Variances of Demand Shocks



Treating these solutions for equilibrium prices and expenditure shares as observed data, we next implement our reverse-weighting estimation procedure from Section 2.2.2, and estimate both the elasticity of substitution across goods ($\hat{\sigma}^{RW}$) and demand for each good in each year ($\hat{\varphi}^{RW}_{kt}$).

In our first quantitative exercise, we examine the properties of the reverse-weighting estimator as we vary the dispersion of the demand shocks (Proposition 3). We assume 100 common goods, a standard deviation of log marginal costs of 1, and a standard deviation of log demand ranging from 0.001 to 1. At the upper tail, this implies that holding fixed prices a one standard deviation movement in shares would cause them to fall 63 percent or rise 170 percent each period. Thus, we span very small and very large demand shocks. In Figure 2, we show the mean and standard deviation of the reverse-weighting estimate across the 250 replications. As the dispersion of the demand shocks declines, the mean estimated elasticity converges to the true value of 4 (top panel), and the standard deviation of the estimated elasticity converges to 0 (bottom panel). Therefore, our reverse-weighting estimator consistently estimates the true parameter value as demand shocks become small as we proved in Proposition 3. What is, perhaps, most surprising is that the mean reverse-weighting estimator remains is within 5 percent of the true parameter value, even when demand shocks are quite large.

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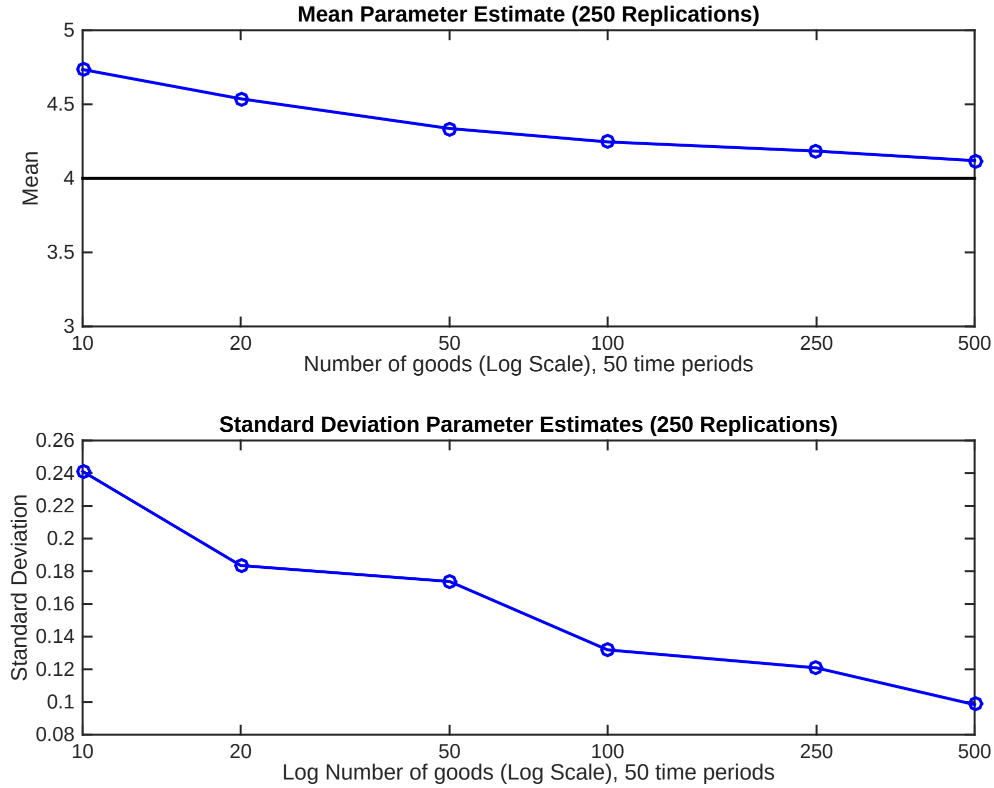
In our second quantitative exercise, we consider the performance of the reverse-weighting estimator as we vary the number of common goods (Proposition 4). We assume standard deviations of log demand and log marginal costs of 1 and vary the number of common goods from 10 to 500. In Figure 3, we show the

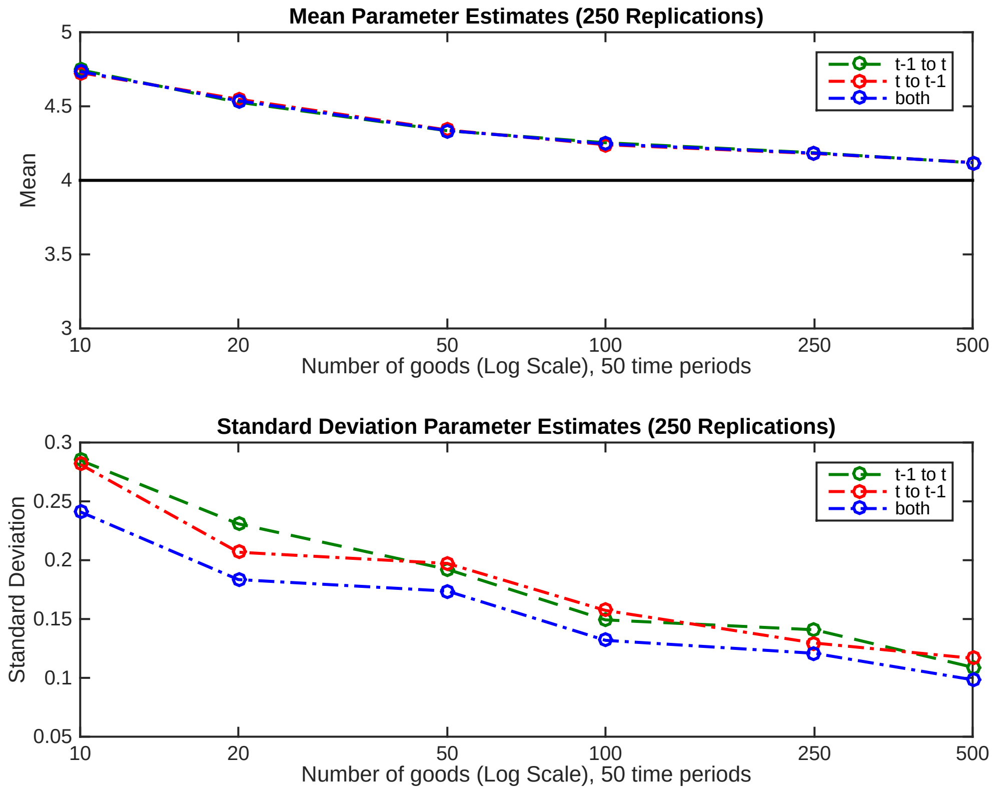
Figure 3: Mean and Standard Deviation of Estimated Elasticity of Substitution for Different Numbers of Common Goods



mean and standard deviation of the reverse-weighting estimate across the 250 replications. As the number of common goods increases, the mean estimated elasticity converges to the true value of 4 (top panel), and the standard deviation of the estimated elasticity converges to 0 (bottom panel). Therefore, in line with the analytical results in Proposition 4, our reverse-weighting GMM estimator consistently estimates the true parameter value as the number of common goods becomes large for independently and identically distributed demand shocks. Furthermore, even for relatively small numbers of common goods (bar-code datasets typically have hundreds or thousands of goods per product group), the mean reverse-weighting estimate remains close to the true parameter value.

In a third quantitative exercise, we report the results of an overidentification check on the reverse-weighting estimator. The moment function (29) for the reverse-weighting estimator includes two elements, the first of which ($m_t^1(\sigma)$) uses the forward difference for the change in the cost of living from period $t-1$ to t (based on period $t-1$ expenditure shares), and the the second of which ($m_t^2(\sigma)$) uses the backward difference for the change in the cost of living from period t to $t-1$ (based on period t expenditure shares). Therefore, we compare estimating the elasticity of substitution using both moment conditions (the reverse-weighting estimator labelled "both") with using only the forward difference moment condition (labelled "$t-1$ to t") and using only the backward difference moment condition (labelled "t" to "$t-1$"). In Figure 4, we report the results assuming standard deviations of log demand and log marginal costs of 1 and numbers of common goods ranging from 10 to 500. We find a similar mean estimated elasticity of substitution (top panel) for all three specifications, but find a lower standard deviation of the estimated elasticity using both

Figure 4: Mean and Standard Deviation of Overidentified and Exactly Identified Estimates of the Elasticity of Substitution for Different Numbers of Common Goods



moment conditions than using only one of the moment conditions (bottom panel). This pattern of results is consistent with the use of both the forward and backward differences imposing the full implications of constant aggregate utility in the model, which is reflected in a gain in efficiency in the overidentified reverse-weighting specification relative to either one of the exactly-identified specifications.

Across all three of these quantitative exercises, we find that the reverse-weighting estimator consistently recovers the model's parameters when the data are generated according to model.

5.2 Data

We will estimate the model using bar-code data from the Nielsen HomeScan database. A major advantage of bar-code data over other types of price and quantity data is that product quality does not vary within a bar code. Thus, changes in physical attributes manifest themselves through the creation (and destruction) of bar-coded goods, not changes in the characteristics of existing bar-coded goods. This property means that changes in sales conditional on price must reflect demand changes not supply changes. Similarly, given that the physical attributes of each bar-coded product is fixed, identification is simpler because supply shocks affect market shares *through* prices, and demand shocks are movements in market share *conditional* on price. Thus, bar-code data is ideal for estimating parameters in our setup.

The bar-code dataset we use is from Kilts-Nielsen and is identical to that in Hottman Redding, and Weinstein (2016), so we just summarize the data description here. The data is based on a sample of approximately 55,000 households who scan in the price and quantity of every bar-coded good they buy.

These households cover a demographically representative sample of households in 42 cities in the United States. The set of goods represents close to the universe of bar-coded goods available in grocery, mass-merchandise, and drug stores, representing around a third of all goods categories in the CPI. We collapse the household and time-dimensions in the data to construct a quarterly sample of total value sold, total quantity sold, and average price. Nielsen organizes the data into close to one hundred "product groups" corresponding to where goods are usually stocked in a store. For example, bar codes within the largest product group "carbonated beverages" are usually grouped together as are bar codes in the next largest product groups of "paper towels," "pet food," and "tobacco."

5.3 Estimates of the Elasticity of Substitution

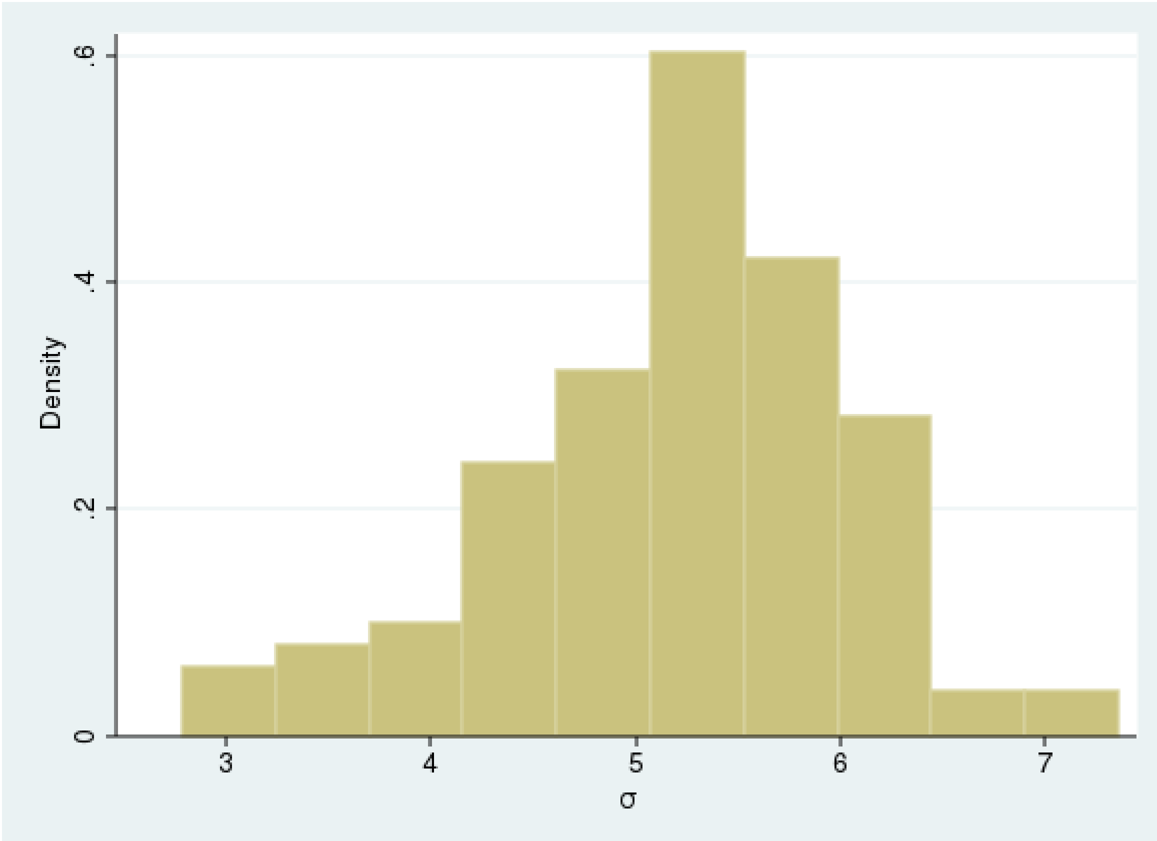
As we have argued earlier, our unified approach differs from statistical indexes, such as the Laspeyres, Paasche, and Jevons. Figure 5 shows the distribution of elasticities of substitution for each product group. The mean and median elasticity of substitution is 5.3, which is substantially larger than the implicit elasticity of 0 in Laspeyres indexes and 1 in the Jevons index. In other words, estimated rates of product substitution based on statistical indexes are likely to dramatically understate the degree of substitution by consumers. In terms of magnitudes, these elasticities do not differ that much from other studies. For example, Hottman, Redding, and Weinstein (2016), using the same data (but a different nesting structure) found that the elasticity of substitution across firms had a median value of 3.9 and 6.9 across firms, so our estimate, which pools within and across firms falls in-between these. Moreover, as one can see in Figure 6, the estimates are precisely estimated—all of them are significantly larger than one.

Estimation is only required in our setup because in the move to discrete time, the forward difference is not equal to the backward difference. In the limit, as time changes become infinitesimal, these two differences become identical, but it is reasonable to ask how different would our estimates be if we used only the moment condition arising from the forward difference or the backward difference. We can denote the elasticity obtained from the reverse-weighting estimator in equation (29) by σ_g^{RW}. Similarly, if we want to focus the information contained in the forward difference, we can obtain a elasticity, σ_g^F, by dropping the second row of the moment vector and only use equations (24) and (26) to obtain our estimate of the elasticity. Similarly, we can only use the information contained in the backward difference by using use equations (25) and (26) and only using the second row of the moment condition in equation (29). Both differences yield extremely similar estimates of the elasticity of substitution. The standard deviation of $\sigma_g^F / \sigma_g^{RW}$ is XX, while that of $\sigma_g^B / \sigma_g^{RW}$ is XX . The similarity of these results means that the degree of substitutability among consumers in either time period is almost identical.

5.4 Comparing Conventional Measures of Price Changes and Welfare

We have already argued that our framework nests many existing methods of measuring price changes and welfare. This nesting makes it possible to step-by-step show how important each assumption is in measuring price changes. In each case, we construct the index for annual differences in price levels for

Figure 5: Distribution of Systems Estimates Across Product Groups



every product group in our sample. With 7 time periods and 87 product groups, we can work with a sample of 609 price changes.

Since the implementations of the Dutot and Carli indexes by statistical agencies render them equivalent to Laspeyres, we will not focus on these indexes. Instead, we will compare the two main indexes of the statistical approach—the Laspeyres and Cobb-Douglas[9]—with superlative indexes (Fisher and Törnqvist) as well as the exact Sato-Vartia CES. This last index is exactly equal to our unified price index under the assumption that product appeal does not vary across time ($\varphi_{kt}/\varphi_{k,t-1} = 1$), and there are no changes in the set of goods across time ($\lambda_{k,t}/\lambda_{k,t-1} = 1$). As we showed in Section 3.2, the Fisher, Törnqvist, and Sato-Vartia CES index are quite similar in that they only differ in how to construct the mean of the expenditure shares in the two time periods.

Figure 7 presents histograms of every 4-quarter price change in our data at the product group level computed for each index. We express each inflation rate as a difference from the superlative Fisher index. The most noticeable feature of the graph is that all of the economic indexes yield almost exactly the same inflation rates on average. The Törnqvist and Sato-Vartia CES typically record an average inflation rate that is identical to the Fisher index up to less than on. Moreover, there is very little dispersion in these indexes. The standard deviation of the inflation rate difference with the Fisher is only 0.1 percentage points per year. We also can replicate this same patter in our Monte Carlo exercise, which demonstrates that the result is not simply a feature of using bar-code data. Thus, if one is willing to ignore new goods and the possibility that

[9] The formulas for these indexes are given in equations (37) and (41).

Figure 6: Estimated Elasticities and Bootstrap Standard Errors

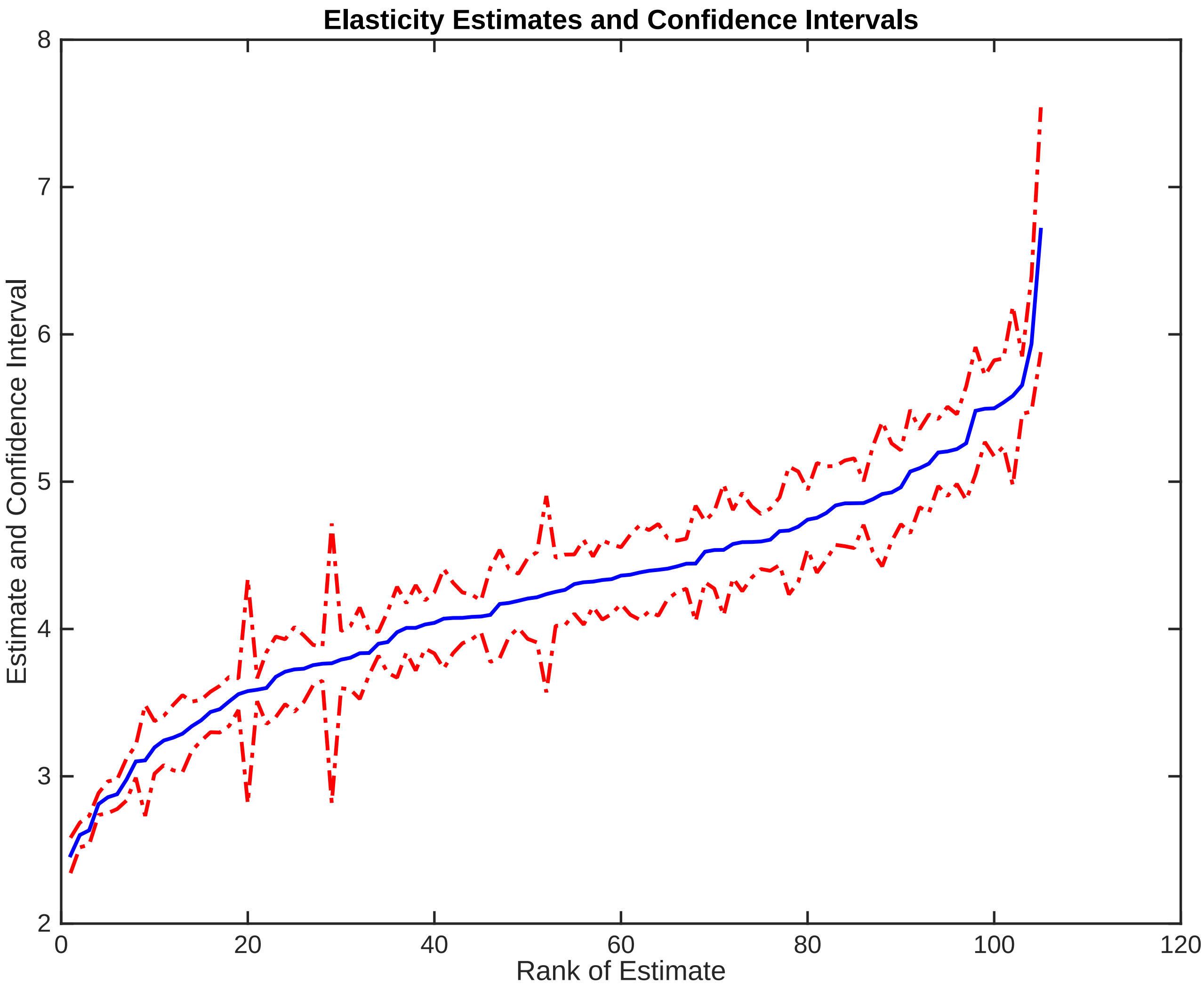
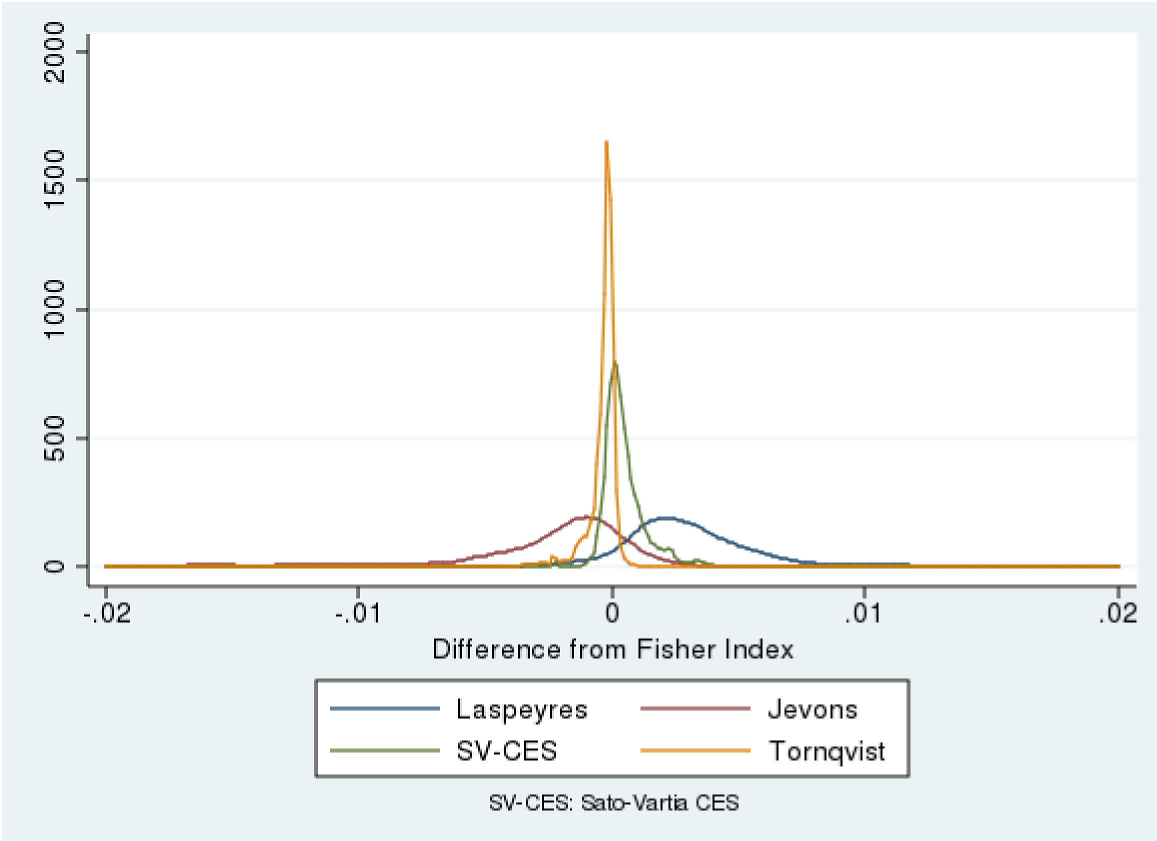


Figure 7: Each Index Differenced from the Superlative Fisher Index



Note: Calculated based on data from The Nielsen Company (US), LLC and provided by the Marketing Data Center at The University of Chicago Booth School of Business.

tastes may be changing, it simply does not matter whether one evaluates price changes using a superlative index or an exact index based on translog, quadratic, or CES preferences.

Figure 7 also reveals the well-known fact that there are biases in price measurement. Exact and Superlative indexes tend to produce lower price changes than the Laspeyres. We can see this fact visually in the Figure 7. Table 1 reports the same information numerically for time periods of different duration. The mean value in the Fisher provides the average ratio in the inflation rate as measured by the Fisher index between $t - \Delta$ and t, where the value of Δ depends on whether we are taking annual, or long differences. Cumulative differences obtained by simply multiplying the annual changes together. For each index below the first panel we report the weighted-mean inflation rate for that index in which we aggregate indexes using $t - \Delta$ expenditure-share weights. In addition, we report the standard deviation and percentile distribution of the *difference* between each index and the Fisher. For example, for the Törnqvist index, we see that the mean inflation rate at an annual frequency is 1.9 percent per year (which is the same as that of the Fisher), and the standard deviation between the Törnqvist and the Fisher across all years and product groups is only 0.1 percentage points. In other words, these indexes are extremely similar both in aggregate and for each product group. More important for our purposes is the fact that the Sato-Vartia CES index, which is equal to the unified price index under the assumption of no demand shifts and no variety changes, is almost identical to the Fisher as well, with an almost identical measured inflation rate. At an annual frequency, the standard deviation of the difference between the Sato-Vartia CES and the Fisher is also only 0.1 percentage points per year. *In other words, any novelty in our unified price index relative to existing economic indexes does not arise from the CES functional form assumption, but rather from our relaxation of the assumptions of constant demand for each good and a constant set of goods.*

We also can replicate standard index number results in our data. The Laspeyres index overstates inflation by around 0.3 to 0.5 percentage points per year depending on the data frequency. The median difference between the Laspeyres inflation rate and the Fisher 0.5 percentage points , and the Jevons lies about the same, which is very close to the 0.34 percentage point per year bias the BLS estimated would be eliminated by moving from a Laspeyres to Jevons index.[10]

Because the unified index nests the Sato-Vartia, Jevons, and Laspeyres and Paasche, we can obtain simple intuition for these results. The Jevons index arises from the CES demand equation under the assumption of a unit elasticity. If instead one focuses on the CES unit expenditure function under the assumption that the elasticity of substitution is zero, one can obtain the Laspeyres index by taking the forward difference and the Paasche by taking the backward difference. While the Sato-Vartia index reconciles all these different approaches (under the assumption of no demand shocks), computing any one of these indexes can be thought of as imposing particular moment conditions on the data. Thus, we should expect to see that the Laspeyres *overstates* inflation relative to the CES and superlative indexes because it does not account for the fact that consumers in the initial period will adjust to future price changes by purchasing cheaper varieties. Similarly, the Paasche index *understates* past price increases because it embodies the assumption that current consumers do not spend much on goods that have risen in price because they do like them, not because they have substituted away from them. The Jevons index, by allowing an elasticity of one, allows for more substitutability and therefore comes much closer to the Sato-Vartia CES and superlative indexes.

There is a second difference related to whether indexes are quadratic means of order r or not. The quadratic mean indexes—Fisher, Törnqvist, and Sato-Vartia—do not just have the same mean inflation rates, but there is almost no dispersion in these inflation rates across sectors. All of these indexes weight price relatives by the average of past and current expenditure shares. For example, the Törnqvist weights the log price changes by an arithmetic average of past and current shares while the Sato-Vartia CES index weights them by a logarithmic average of the two shares. While these different quadratic means of the same numbers are not identical in theory, in practice they end up being quite similar. However, indexes such as the Jevons, Laspeyres, and Paasche are not quadratic means of the prices and therefore have much more dispersion in the implied inflation rates depending on how the shares move. These differences are quite volatile with standard deviations ranging from 0.3 percentage points for the Laspeyres and 0.7 for the Jevons.

Another way of understanding these results is to look at the percentile distributions of the difference between each index and the Fisher. In order to investigate this, we compute the value of each index for each product group and subtract the value of the Fisher index using the same data. We then report the standard deviation and the 5th, 25th, median, 75th, and 95th percentiles of these differences. One testament to how similar the the quadratic mean indexes—Fisher, Törnqvist and Sato-Vartia—are is the fact that the

[10] See, http://www.bls.gov/cpi/cpigmrp.htm

1

Table 1:

Fisher	Annual	Cumulative, 2004-2011	Long, 2004-2007	Cumulative, 2004-2007
Mean	1.9	14.1	6.7	5.3
Standard Deviation	3.9	4.5	5.6	4.0
5th Percentile	-3.4	-3.4	-4.1	-2.9
50th Percentile	1.0	1.4	4.3	1.2
95th Percentile	7.6	8.1	14.3	6.8
Tornqvist				
Mean	1.9	13.8	6.6	5.3
Standard Deviation	0.1	0.1	0.2	0.1
5th Percentile	-0.2	-0.1	-0.7	-0.1
50th Percentile	-0.0	-0.0	-0.1	-0.0
95th Percentile	0.0	0.0	-0.0	0.0
Sato-Vartia CES				
Mean	1.9	14.4	6.7	5.5
Standard Deviation	0.1	0.1	0.2	0.1
5th Percentile	-0.0	-0.0	-0.1	-0.0
50th Percentile	0.0	0.0	0.1	0.0
95th Percentile	0.2	0.2	0.6	0.2
Jevons				
Mean	1.7	12.2	6.2	4.6
Standard Deviation	0.7	0.6	1.5	0.5
5th Percentile	-1.9	-1.6	-4.2	-1.3
50th Percentile	-0.2	-0.2	-0.3	-0.2
95th Percentile	0.1	0.1	0.2	0.1
Laspeyres				
Mean	2.2	16.4	7.1	6.2
Standard Deviation	0.3	0.3	0.7	0.3
5th Percentile	-0.1	-0.1	-0.4	-0.1
50th Percentile	0.3	0.3	0.5	0.2
95th Percentile	1.1	0.8	1.7	0.8
CG-UPI				
Mean	2.0	14.4	9.8	9.2
Standard Deviation	8.1	12.0	33.2	10.9
5th Percentile	-8.1	-8.1	-16.2	-6.4
50th Percentile	-0.8	-0.7	0.9	0.2
95th Percentile	6.5	8.3	15.4	8.7
UPI				
Mean	-0.8	-5.6	0.3	0.8
Standard Deviation	9.0	8.9	24.0	7.9
5th Percentile	-15.8	-14.7	-34.1	-9.6
50th Percentile	-2.9	-2.7	-6.6	-1.7
95th Percentile	3.4	4.5	3.1	5.0

Calculations on data from The Nielsen Company (US), LLC and provided by the Marketing Data Center at The University of Chicago Booth School of Business.

The mean is the past-period expenditure share weighted mean of the index and the standard deviation and percentiles for all indexes except the Fiisher are computed based on the difference between the index and the Fisher index at the product group level. In the case of the Fisher index, the standard deviation and percentiles and percentiles correspond to the actual variation of the product-group indexes.

Note: The mean is the past-period expenditure share weighted mean of the index and the standard deviation and percentiles for all indexes except the Fisher are computed based on the difference between the index and the Fisher index at the product group level. In the case of the Fisher index, the standard deviation and percentiles and percentiles correspond to the actual variation of the product-group indexes. Calculated based on data from The Nielsen Company (US), LLC and provided by the Marketing Data Center at The University of Chicago Booth School of Business.

standard deviation of the differences between them are generally quite small. In other words, it does not matter much for price measurement which of these indexes are used. The same cannot be said for the Jevons and Laspeyres indexes which have standard deviations of dispersion from the Fisher indexes that are typically much larger at the annual frequency. For example, ninety percent of the time the Jevons index is between 1.9 percentage points smaller than the Fisher or 0.1 percentage points larger. In other words, it's only accurate to around 2 percentage points at an annual frequency.

5.5 Variable Demand and Price Indexes

The previous section showed that under standard assumptions all exact and superlative indexes return almost identical measures of price changes when used on bar-code data. As one can see from Figure 1, all major extant indexes assume that demand is constant and are therefore inconsistent with the demand system, while the unified approach allows product appeal to be time varying, which allows a demand system with an error term. We first need to establish that demand shocks exist. We can establish this in a straightforward way. If we first difference equation (4) and replace the price index term with a time dummy, we obtain the following equation:

$$\Delta \ln S_{kt} = \alpha_t + \beta \Delta \ln P_{kt} + \epsilon_{kt} \tag{56}$$

where α_t is a time varying dummy that captures the impact of the price index on market shares, β is a parameter, and ϵ_{kt} is an error that equals the elasticity multiplied by the demand shock: $(\sigma - 1)\,\Delta \ln \varphi_{kt}$. As long as prices are correlated with demand $\hat{\beta}$ will not equal $(1-\sigma)$, but $\left(1 - R^2\right)$ provides a lower bound for the importance of demand shocks. As long as the R^2 is below one, we know that demand shocks must exist in our data, violating a key assumption underlying all economically-motivated price indexes. Not surprisingly, when the run this regression at the product-group level, allowing all the regression parameters to vary by product group, we obtain a median R^2 of 0.11. The fact that this R^2 is below one establishes that demand shocks are a feature of our data. Moreover, the low R^2 establishes that demand shocks must play an important role in understanding consumer behavior.

The fact that demand shocks exist raises the question of how much they matter for understanding utility. As we showed in the proof to proposition 9, we can use the demand equation to solve for elasticity of substitution based on the Sato-Vartia formula according to equation (104) in the case of no demand shocks. If demand shocks for understanding the utility function, we would expect this key utility parameter to be stable as well. In order to operationalize how demand shocks affect the implied elasticity of substitution, we denote the implied Sato-Vartia elasticity of substitution for each period by σ_{gt}^{SV} for every 4-quarter difference and product-group. Obviously, we should expect these estimates to vary by product group, so we are interested in the dispersion of these estimates relative to the product group mean, or $\left(\sigma_{gt}^{SV} - \frac{1}{T}\sum_t \sigma_{gt}^{SV}\right)$, where T is the number of periods. If demand shocks are zero, we should expect this number to be close to zero.

Table 2 presents the mean of $\frac{1}{T}\sum_t \sigma_{gt}^{SV}$, and the distribution of $\left(\sigma_{gt}^{SV} - \frac{1}{T}\sum_t \sigma_{gt}^{SV}\right)$. The mean value is negative , which as one can see is -0.65. which means that obtaining elasticities (and price indexes) under

Table 2: $\sigma_{gt}^{SV} - \frac{1}{T}\sum_t \sigma_{gt}^{SV}$

	Mean	Standard Deviation	1st Percentile	5th Percentile	25th Percentile	50th Percentile	75th Percentile	95th Percentile	99th Percentile
Difference Between Elasticities and the Product Group Mean	-0.65	53.03	-104.58	-28.20	-4.37	-0.11	5.40	33.33	168.79

Calculations on data from The Nielsen Company (US), LLC and provided by the Marketing Data Center at The University of Chicago Booth School of Business.

Note: The standard deviation and percentiles report the the deviations of the Sato-Vartia elasticity from the average value for each product group. Calculated based on data from The Nielsen Company (US), LLC and provided by the Marketing Data Center at The University of Chicago Booth School of Business. The mean is the average of all elasticities of substation at the product-group level computed using equation (104).

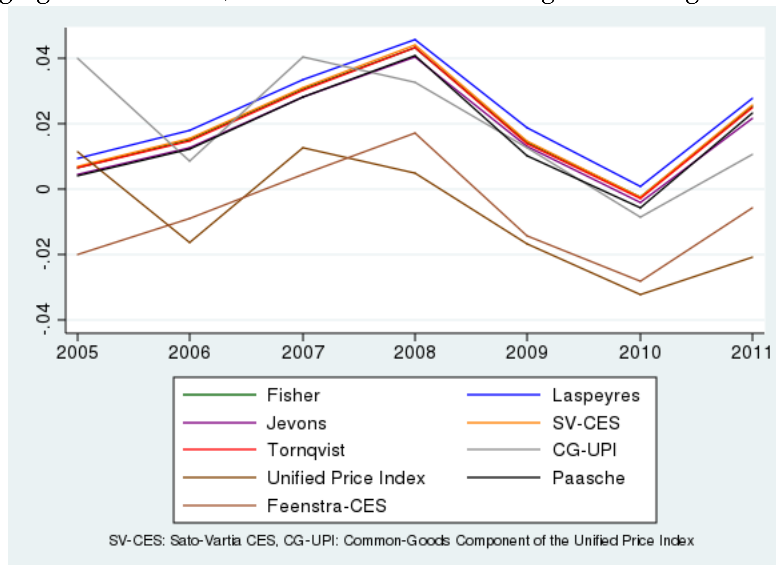
the assumption of no demand shocks make little economic sense. Moreover, there is a tremendous variation in the implied elasticities. As we look at the implied elasticities *within a product group* and across years, we see that half of all observations entail movements outside the range of 4.4 below the average implied elasticity to 5.4 above it. This enormous variation in the implied values of the elasticity of substitution, which spans all reasonable and many unreasonable values, means that the Sato-Vartia formula is a deeply flawed way of thinking about price movements. If one believes the underlying assumption of the exact price index—that demand is constant—then one should also believe that the utility function itself is varying substantially. However, if the utility function is not constant across more than one time difference, it is difficult to give any economic interpretation for what the price index is measuring.

5.6 The Unified Price Index

As we showed in Proposition 9, the Sato-Vartia Exact CES price index is only correct if demand shocks are zero, whereas the unified index is correct even in the presence of demand shocks. An obvious question is how much this matters for price measurement. Table 1 compares how the common-goods component of the unified price index (the term in square brackets in equation (21) differs from the Fisher and similarly, the Sato-Vartia. The good news is that the average inflation rate is quite similar: 1.9 percent for the Sato-Vartia and 2.0 percent for the Fisher. However, there is tremendous dispersion in the two indexes. The standard deviation of the difference between the two indexes is 8.1 percentage points, which means that there are substantial differences in how the two indexes measure inflation in a particular product group.

We can see these differences at the aggregate level in Figure 8, which plots the aggregate inflation rates computed using the same data but using the different formulas. Not surprisingly, the Fisher, Törnqvist, and Sato-Vartia indexes result in almost identical inflation rates that are bounded by the Paasche and Laspeyres indexes. While the common-goods unified price index differs very little on average from these indexes, there is a lot of year-to-year variation, sometimes exceeding three percentage points. The second column of Table 1, however, reveals that these demand shocks tend to cancel over long time periods. The Sato-Vartia and CG-UPI, for example, both yield seven-year inflation rates of 14.4 percentage points. In other words, whether one allows demand to vary or not matters enormously for understanding annual inflation rates, but they tend to cancel over long horizons.

Figure 8: Aggregate Price Index, Calculated as a Share-Weighted Average Price Growth Rate



Note: Calculated based on data from The Nielsen Company (US), LLC and provided by the Marketing Data Center at The University of Chicago Booth School of Business.

Table 3: Correlation of Price Changes (Product-Group-Year)

Variables	Fisher	Tornqvist	SV-CES	Jevons	Laspeyres	CG-UPI	Feenstra-CES
Tornqvist	0.9999						
SV-CES	0.9998	0.9997					
Jevons	0.9931	0.9942	0.9918				
Laspeyres	0.9973	0.9975	0.9972	0.9931			
CG-UPI	0.0230	0.0225	0.0254	0.0029	0.0084		
Feenstra-CES	0.8699	0.8724	0.8650	0.8950	0.8665	-0.1011	
UPI	0.2658	0.2673	0.2638	0.2707	0.2497	0.8844	0.3047

Note: Calculated based on data from The Nielsen Company (US), LLC and provided by the Marketing Data Center at The University of Chicago Booth School of Business.

Table 4: Correlation of Price Changes (Annual Aggregate)

Variables	Fisher	Tornqvist	SV-CES	Jevons	Laspeyres	CG-UPI	Feenstra-CES
Tornqvist	1.0000						
SV-CES	1.0000	1.0000					
Jevons	0.9991	0.9991	0.9991				
Laspeyres	0.9992	0.9991	0.9992	0.9996			
CG-UPI	0.5448	0.5470	0.5474	0.5549	0.5389		
Feenstra-CES	0.9892	0.9895	0.9891	0.9911	0.9878	0.5610	
UPI	0.5099	0.5123	0.5123	0.5225	0.5034	0.9900	0.5457

Note: Calculated based on data from The Nielsen Company (US), LLC and provided by the Marketing Data Center at The University of Chicago Booth School of Business.

We can see the problem of how to measure inflation rates at an annual frequency even more clearly in Tables 3 and 4, which present the correlation coefficients between each of our indexes at the product-group level and at the aggregate level. Standard price indexes are highly correlated amongst themselves with correlation coefficients of 0.997 or higher. However, at an aggregate level the correlation between the common-goods component of the UPI and standard indexes drops to around 0.5 at an annual frequency. Moreover, at the product-group level these indexes the CG-UPI is almost uncorrelated with standard indexes. This result means that even at an aggregate level, whether one believes demand shocks exist is of first-order importance in price measurement.

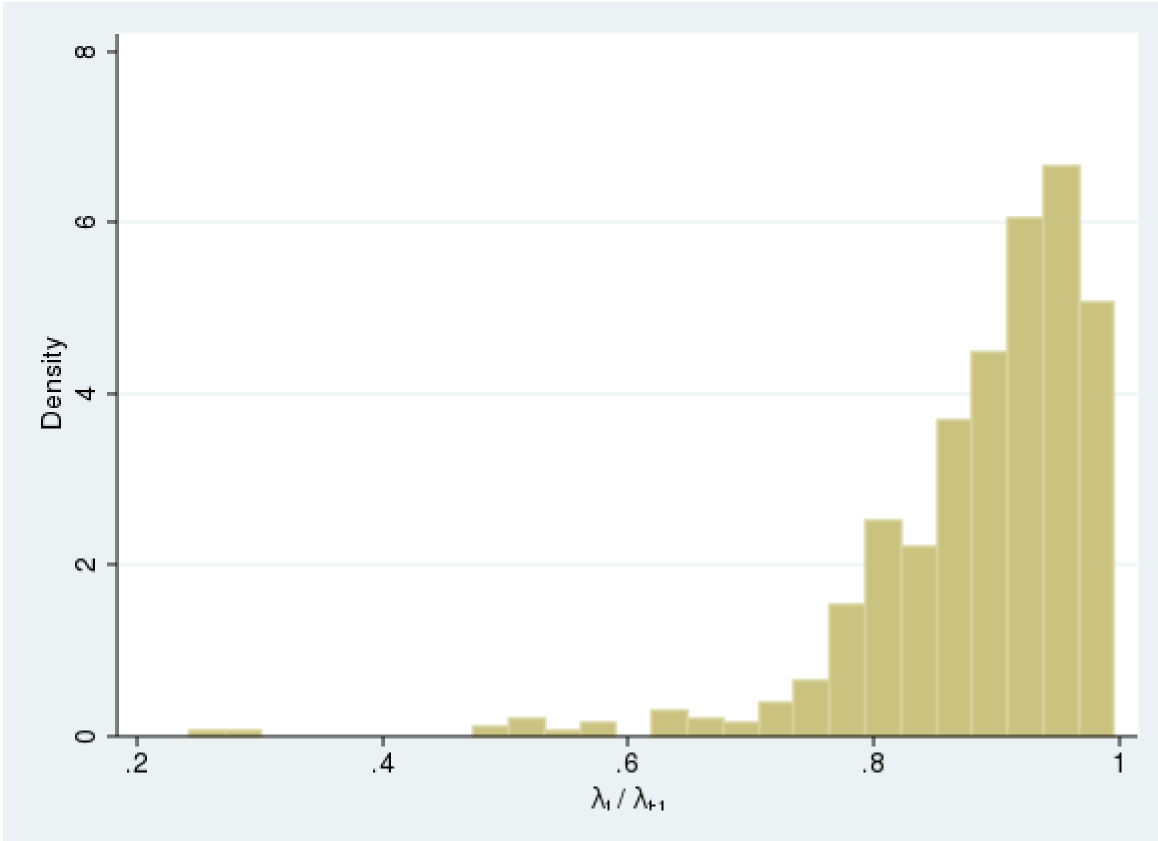
The result that the CG-UPI is quite similar to the Sato-Vartia CES index and superlative long run does not imply that there are not important differences. The result is very dependent on averaging across many years and product groups. As one can see in Table 1, the picture for individual product groups is much messier. Over a seven-year horizons the standard deviation of the differences between the CG-UPI and the Fisher is 12 percentage points. Thus, while it is true that when one averages across time and product groups (609 observations) the law of large numbers results in small errors on average, the problems are still large when measuring prices of individual industries (product groups), which only consist of seven years of data. A related point is that the demand shocks do not tend to cancel unless one observes price changes over a relative long period. Columns 3 and 4 of Table 1 report long and cumulative growth rates for three year periods. In these cases, the differences between the CG-UPI and the superlative indexes or Sato-Vartia are quite substantial (around 4 percentage points).

These differences are even larger if on compares the CG-UPI with the indexes actually used by statistical agencies. The two relevant indexes for how prices are actually measured in official statistics are the Jevons and Laspeyres. For example, the US measures prices with the Jevons and Japan uses the Laspeyres (based on the Dutot). Given this different measurement approach, our estimates imply that over a seven-year period, the US will underreport inflation by 2.2 percentage points and Japan will overstate inflation by 2 percentage points. Thus, there will be a 4.2 percentage point differential in measured welfare over just seven years just due to what assumptions were used to build the price indexes. These differences are quite large relative to standard explanations of real growth differentials like differences in productivity growth. Our estimates imply that different assumptions used in constructing price differences are likely to be an important factor in understanding differential long-run growth rates.

Adding in the variety-adjustment term in equation (32) moves us from the common-goods component of the UPI to the unified price index. This term combines the elasticity of substitution, which tells us how much consumers value varieties, with the rates of product creation and destruction. Figure 9 presents a histogram of the λ_t/λ_{t-1} ratios that drive the variety bias. Not surprisingly, most of these λ_t/λ_{t-1} ratios are less than one indicating that new goods tend to have lower price-to-appeal ratios than disappearing ones. As one can see in Table 1, the relatively rapid rate of new good creation results in the mean and median unified-index price increase across all product groups and times being 2.8 percentage points lower than the common-goods component of the unified index and 2.7 percentage points below the Sato-Vartia.

As one can see from Figure 8, both the UPI and the Feenstra-CES index lie substantially below standard indexes because standard indexes do not take product turnover into account. What is more novel, however,

Figure 9: $\lambda_t / \lambda_{t-1}$, Annual Differences



Note: Calculated based on data from The Nielsen Company (US), LLC and provided by the Marketing Data Center at The University of Chicago Booth School of Business.

is how different these indexes are from conventional indexes and each other. Tables 3 and 4 report the correlation coefficients between all of the indexes. Like its common-goods component, the UPI only has a correlation coefficient of 0.5 with standard indexes at the aggregate level and is close to uncorrelated at a disaggregrate level. Interestingly, this result appears to be entirely driven by the fact that the UPI allows for demand shifts while the other indexes do not. This fact also explains why the Feenstra-CES index is much more highly correlated with exact and superlative indexes than the UPI—despite the same variety adjustment as the UPI, the Feenstra index uses the Sato-Vartia index to compute common goods price changes. The low correlation between the UPI and other indexes however, suggests that much of what we measure as annual price changes is critically dependent on what implicit assumptions we use to approach the data, not on the data itself.

One of the most striking differences in this table is between the UPI and the Sato-Vartia CES and superlative indexes. While all of these indexes the UPI is quite different. Over the seven year period the aggregate CG-UPI *rose* by 14.4 percent, while the UPI *fell* by 5.6 percent. There was a 20 percentage point gap! In other words, whether one thinks prices rose or fell between 2004 and 2011 depends crucially on whether one includes data on product turnover. Moreover, as one can see in Figure 8 and Table 4, the Feenstra-CES and the UPI are not very correlated at the annual frequency because the former index is based on the Sato-Vartia CES index. Clearly, the assumptions used to construct price indexes matter enormously.

6 Conclusions

We have developed a unified price index that provides a consistent way of estimating demand systems and measuring welfare changes. Our index nests all major approaches to measuring aggregate price changes. Existing indexes can be derived from the UPI by ignoring data, moment conditions, and/or imposing particular parameter restrictions. In this sense, existing approaches to welfare measurement can be thought of as special cases of the UPI. Our unified approach also resolves inconsistencies in the measurement of demand and aggregate price changes. All exact and superlative indexes assume time-invariant demand, whereas all demand system estimation assumes demand shifts. The unified price index provides a means of consistently estimating demand and aggregate price movements.

At a deep level, our paper demonstrates a paradox in economic measurement. Aggregate nominal variables, such as sales or expenditures, are typically observable, and therefore their measurement is uncontroversial. The measurement of aggregate real variables, such as welfare or real output, can only be done using theoretical constructs such as price indexes; and price indexes implicitly impose a preference structure on the data. In other words, the problem of economic measurement is that nominal variables are based on *facts*, and real variables are based on *ideas*. Sometimes these ideas are vague as in the three main statistical indexes used in measurement—Dutot, Carli, and Jevons—but our work shows that these are just restricted versions of our more general theory. Similarly, the upward bias of the Laspeyres and the downward bias of the Paasche only make sense if one does not believe a *model* of consumer behavior in which the elasticity of substitution is zero.

These ideas matter enormously for how one interprets price and welfare movements. Aggregate price movements at an annual frequency are extremely sensitive to precisely whether one believes that demand shocks are nonexistent, elasticities equal zero or one, and the creation of new goods is irrelevant. By relaxing these assumptions and conditions, the unified price index produces very different measures of aggregate price changes than conventional indexes, even over long time periods.

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7 Appendix

7.1 Equivalent Expressions for the Change in the Price Index

We begin by reproducing the expression for the change in the firm price index going forward in time from period $t-1$ to t (13):

$$\Phi_{t-1,t} = \frac{\mathbb{P}_t}{\mathbb{P}_{t-1}} = \left[\frac{\sum_{k\in\Omega_t} \left(P_{kt}/\varphi_{kt}\right)^{1-\sigma}}{\sum_{k\in\Omega_{t-1}} \left(P_{kt-1}/\varphi_{kt-1}\right)^{1-\sigma}} \right]^{\frac{1}{1-\sigma}}. \tag{57}$$

Derivation of $\Phi^F_{t-1,t}$ in (22): Multiplying the numerator and denominator of (57) by the summation $\sum_{k\in\Omega_{t,t-1}} \left(P_{kt}/\varphi_{kt}\right)^{1-\sigma}$ over common goods at time t, we obtain:

$$\Phi^F_{t-1,t} = \left[\frac{\sum_{k\in\Omega_t} \left(P_{kt}/\varphi_{kt}\right)^{1-\sigma}}{\sum_{k\in\Omega_{t,t-1}} \left(P_{kt}/\varphi_{kt}\right)^{1-\sigma}} \frac{\sum_{k\in\Omega_{t,t-1}} \left(P_{kt}/\varphi_{kt}\right)^{1-\sigma}}{\sum_{k\in\Omega_{t-1}} \left(P_{kt-1}/\varphi_{kt-1}\right)^{1-\sigma}} \right]^{\frac{1}{1-\sigma}},$$

which using (14) can be re-written as:

$$\Phi^F_{t-1,t} = \left[\frac{1}{\lambda_t} \frac{\sum_{k\in\Omega_{t,t-1}} \left(P_{kt}/\varphi_{kt}\right)^{1-\sigma}}{\sum_{k\in\Omega_{t-1}} \left(P_{kt-1}/\varphi_{kt-1}\right)^{1-\sigma}} \right]^{\frac{1}{1-\sigma}}.$$

Multiplying the numerator and denominator by the summation $\sum_{k\in\Omega_{t,t-1}} \left(P_{kt-1}/\varphi_{kt-1}\right)^{1-\sigma}$ over common goods at time $t-1$, we have:

$$\Phi^F_{t-1,t} = \left[\frac{1}{\lambda_t} \frac{\sum_{k\in\Omega_{t,t-1}} \left(P_{kt-1}/\varphi_{kt-1}\right)^{1-\sigma}}{\sum_{k\in\Omega_{t-1}} \left(P_{kt-1}/\varphi_{kt-1}\right)^{1-\sigma}} \frac{\sum_{k\in\Omega_{t,t-1}} \left(P_{kt}/\varphi_{kt}\right)^{1-\sigma}}{\sum_{k\in\Omega_{t,t-1}} \left(P_{kt-1}/\varphi_{kt-1}\right)^{1-\sigma}} \right]^{\frac{1}{1-\sigma}},$$

which using the share of expenditure on the set of common goods (14) can be expressed as:

$$\Phi^F_{t-1,t} = \left[\frac{\lambda_{t-1}}{\lambda_t} \frac{\sum_{k\in\Omega_{t,t-1}} \left(P_{kt}/\varphi_{kt}\right)^{1-\sigma}}{\sum_{k\in\Omega_{t,t-1}} \left(P_{kt-1}/\varphi_{kt-1}\right)^{1-\sigma}} \right]^{\frac{1}{1-\sigma}} = \left(\frac{\lambda_t}{\lambda_{t-1}} \right)^{\frac{1}{\sigma-1}} \frac{\mathbb{P}^*_t}{\mathbb{P}^*_{t-1}}, \tag{58}$$

which using the share of each good in expenditure on the set of common goods (18) at time $t-1$ becomes:

$$\Phi^F_{t-1,t} = \left[\frac{\lambda_{t-1}}{\lambda_t} \sum_{k\in\Omega_{t,t-1}} \left[\frac{\left(P_{kt}/\varphi_{kt}\right)^{1-\sigma}}{\sum_{k\in\Omega_{t,t-1}} \left(P_{kt-1}/\varphi_{kt-1}\right)^{1-\sigma}} \right] \right]^{\frac{1}{1-\sigma}},$$

$$\Phi^F_{t-1,t} = \left[\frac{\lambda_{t-1}}{\lambda_t} \sum_{k\in\Omega_{t,t-1}} \left[\frac{\left(P_{kt}/\varphi_{kt}\right)^{1-\sigma}}{\frac{1}{S^*_{kt-1}} \left(P_{kt-1}/\varphi_{kt-1}\right)^{1-\sigma}} \right] \right]^{\frac{1}{1-\sigma}},$$

$$\Phi^F_{t-1,t} = \left(\frac{\lambda_t}{\lambda_{t-1}} \right)^{\frac{1}{\sigma-1}} \left[\sum_{k\in\Omega_{t,t-1}} S^*_{kt-1} \left(\frac{P_{kt}/\varphi_{kt}}{P_{kt-1}/\varphi_{kt-1}} \right)^{1-\sigma} \right]^{\frac{1}{1-\sigma}}, \tag{59}$$

and corresponds to equation (22) in the main text above.

Derivation of $\Phi^B_{t,t-1}$ in (23): From equation (58), the change in the price index going backwards in time from period t to period $t-1$ can be re-written as follows:

$$\Phi^B_{t,t-1} = \frac{\mathbb{P}_{t-1}}{\mathbb{P}_t} = \left[\frac{\lambda_t}{\lambda_{t-1}} \frac{\sum_{k\in\Omega_{t,t-1}} \left(P_{kt-1}/\varphi_{kt-1}\right)^{1-\sigma}}{\sum_{k\in\Omega_{t,t-1}} \left(P_{kt}/\varphi_{kt}\right)^{1-\sigma}} \right]^{\frac{1}{1-\sigma}},$$

$$\Phi^B_{t,t-1} = \left[\frac{\lambda_t}{\lambda_{t-1}} \sum_{k\in\Omega_{t,t-1}} \frac{\left(P_{kt-1}/\varphi_{kt-1}\right)^{1-\sigma}}{\sum_{k\in\Omega_{t,t-1}} \left(P_{kt}/\varphi_{kt}\right)^{1-\sigma}} \right]^{\frac{1}{1-\sigma}},$$

$$\Phi^B_{t,t-1} = \left[\frac{\lambda_t}{\lambda_{t-1}} \sum_{k\in\Omega_{t,t-1}} \frac{\left(P_{kt-1}/\varphi_{kt-1}\right)^{1-\sigma}}{\frac{1}{S^*_{kt}} \left(P_{kt}/\varphi_{kt}\right)^{1-\sigma}} \right]^{\frac{1}{1-\sigma}},$$

$$\Phi^B_{t,t-1} = \left(\frac{\lambda_{t-1}}{\lambda_t} \right)^{\frac{1}{\sigma-1}} \left[\sum_{k\in\Omega_{t,t-1}} S^*_{kt} \left(\frac{P_{kt-1}/\varphi_{kt-1}}{P_{kt}/\varphi_{kt}} \right)^{1-\sigma} \right]^{\frac{1}{1-\sigma}}, \tag{60}$$

which corresponds to (23) in the main text above.

Derivation of $\Phi^D_{t-1,t}$ in (21): Using the share of each good in expenditure on the set of common goods (17) at times t and $t-1$, the change in the price index going forward in time from period $t-1$ to period t (58) also can be expressed as:

$$\Phi^D_{t-1,t} = \left(\frac{\lambda_t}{\lambda_{t-1}} \right)^{\frac{1}{\sigma-1}} \frac{P_{\ell t}/\varphi_{\ell t}}{P_{\ell t-1}/\varphi_{\ell t-1}} \left(\frac{S^*_{\ell t}}{S^*_{\ell t-1}} \right)^{\frac{1}{\sigma-1}}, \tag{61}$$

Taking logs of both sides we have:

$$\ln \Phi^D_{t-1,t} = \frac{1}{\sigma-1} \ln \left(\frac{\lambda_t}{\lambda_{t-1}} \right) + \ln \left(\frac{P_{\ell t}}{P_{\ell t-1}} \right) - \ln \left(\frac{\varphi_{\ell t}}{\varphi_{\ell t-1}} \right) + \frac{1}{\sigma-1} \ln \left(\frac{S^*_{\ell t}}{S^*_{\ell t-1}} \right). \tag{62}$$

Taking means of both sides across the set of common goods, we obtain:

$$\ln \Phi^D_{t-1,t} = \frac{1}{\sigma-1} \ln \left(\frac{\lambda_t}{\lambda_{t-1}} \right) + \frac{1}{N_{t,t-1}} \sum_{k\in\Omega_{t,t-1}} \ln \left(\frac{P_{kt}}{P_{kt-1}} \right) - \frac{1}{N_{t,t-1}} \sum_{k\in\Omega_{t,t-1}} \ln \left(\frac{\varphi_{kt}}{\varphi_{kt-1}} \right), \tag{63}$$
$$+ \frac{1}{\sigma-1} \frac{1}{N_{t,t-1}} \sum_{k\in\Omega_{t,t-1}} \ln \left(\frac{S^*_{kt}}{S^*_{kt-1}} \right),$$

where an analogous expression to (63) would also hold if we replaced the weights $1/N_{t,t-1}$ with any arbitrary set of positive constants summing to one, as noted in Banerjee (1983). Rewriting (63), we obtain:

$$\Phi^D_{t-1,t} = \left(\frac{\lambda_t}{\lambda_{t-1}} \right)^{\frac{1}{\sigma-1}} \left(\prod_{k\in\Omega_{t,t-1}} \frac{P_{kt}}{P_{kt-1}} \right)^{\frac{1}{N_{t,t-1}}} \left(\prod_{k\in\Omega_{t,t-1}} \frac{S^*_{kt}}{S^*_{kt-1}} \right)^{\frac{1}{(\sigma-1)N_{t,t-1}}} \left(\prod_{k\in\Omega_{t,t-1}} \frac{\varphi_{kt-1}}{\varphi_{kt}} \right)^{\frac{1}{N_{t,t-1}}}, \tag{64}$$
$$= \left(\frac{\lambda_t}{\lambda_{t-1}} \right)^{\frac{1}{\sigma-1}} \frac{\tilde{P}^*_t}{\tilde{P}^*_{t-1}} \left(\frac{\tilde{S}^*_t}{\tilde{S}^*_{t-1}} \right)^{\frac{1}{\sigma-1}} \frac{\tilde{\varphi}^*_{t-1}}{\tilde{\varphi}^*_t},$$

which corresponds to (21) in the main text above.

7.2 Derivation of Equation (24)

Proof. The first two expressions for the change in the cost of living (22)-(23) can be written as:

$$\frac{\mathbb{P}^*_t}{\mathbb{P}^*_{t-1}} = \left[\Theta_{t-1,t} \sum_{k\in\Omega_{t,t-1}} S^*_{kt-1} \left(\frac{P_{kt}}{P_{kt-1}} \right)^{1-\sigma} \right]^{\frac{1}{1-\sigma}},$$

$$\frac{\mathbb{P}^*_t}{\mathbb{P}^*_{t-1}} = \left[\Theta_{t,t-1} \sum_{k\in\Omega_{t,t-1}} S^*_{kt} \left(\frac{P_{kt}}{P_{kt-1}} \right)^{-(1-\sigma)} \right]^{-\frac{1}{1-\sigma}},$$

where $\Theta_{t-1,t}$ and $\Theta_{t,t-1}$ are aggregate demand shifters:

$$\Theta_{t-1,t} = \frac{\sum_{k\in\Omega_{t,t-1}} S^*_{kt-1} \left(\frac{P_{kt}}{P_{kt-1}}\right)^{1-\sigma} \left(\frac{\varphi_{kt}}{\varphi_{kt-1}}\right)^{\sigma-1}}{\sum_{k\in\Omega_{t,t-1}} S^*_{kt-1} \left(\frac{P_{kt}}{P_{kt-1}}\right)^{1-\sigma}}, \qquad \Theta_{t,t-1} = \frac{\sum_{k\in\Omega_{t,t-1}} S^*_{kt} \left(\frac{P_{kt-1}}{P_{kt}}\right)^{1-\sigma} \left(\frac{\varphi_{kt-1}}{\varphi_{kt}}\right)^{\sigma-1}}{\sum_{k\in\Omega_{t,t-1}} S^*_{kt} \left(\frac{P_{kt-1}}{P_{kt}}\right)^{1-\sigma}}. \tag{65}$$

Now note the following results:

$$S^*_{kt} = \frac{\left(P_{kt}/\varphi_{kt}\right)^{1-\sigma}}{P_t^{1-\sigma}},$$

which implies:

$$\left(\frac{P_{kt}/\varphi_{kt}}{P_{kt-1}/\varphi_{kt-1}}\right)^{1-\sigma} = \frac{S^*_{kt}}{S^*_{kt-1}} \left(\frac{P_t}{P_{t-1}}\right)^{1-\sigma},$$

$$\left(\frac{P_{kt}/\varphi_{kt}}{P_{kt-1}/\varphi_{kt-1}}\right)^{-(1-\sigma)} = \frac{S^*_{kt-1}}{S^*_{kt}} \left(\frac{P_{t-1}}{P_t}\right)^{1-\sigma},$$

$$\left(\frac{P_{kt}}{P_{kt-1}}\right)^{1-\sigma} = \frac{S^*_{kt}}{S^*_{kt-1}} \left(\frac{P_t}{P_{t-1}}\right)^{1-\sigma} \left(\frac{\varphi_{kt}}{\varphi_{kt-1}}\right)^{(1-\sigma)},$$

$$\left(\frac{P_{kt}}{P_{kt-1}}\right)^{-(1-\sigma)} = \frac{S^*_{kt-1}}{S^*_{kt}} \left(\frac{P_{t-1}}{P_t}\right)^{1-\sigma} \left(\frac{\varphi_{kt}}{\varphi_{kt-1}}\right)^{-(1-\sigma)}.$$

Substituting these results into the fractions on the two sides of the equality (65), cancelling terms and noting that $\sum_{k\in\Omega_{t,t-1}} S^*_{kt} = \sum_{k\in\Omega_{t,t-1}} S^*_{kt-1} = 1$, we obtain equation (24) in the main text. □

7.3 Proof of Proposition 2 (Identification)

Proof. Given the data on prices and expenditure shares $\{P_{kt}, S_{kt}\}$, we can construct common goods expenditure shares $\{S^*_{kt}\}$. Under the identifying assumption of constant aggregate CES preferences (using equations (24) and (26) and $\Theta_{t-1,t}^{1/(1-\sigma)} = 1$), we have:

$$\left[\sum_{k\in\Omega_{t,t-1}} S^*_{kt-1} \left(\frac{P_{kt-1}}{P_{kt}}\right)^{\sigma-1}\right]^{-\frac{1}{\sigma-1}} = \left[\frac{\tilde{P}^*_t}{\tilde{P}^*_{t-1}} \left(\frac{\tilde{S}^*_t}{\tilde{S}^*_{t-1}}\right)^{\frac{1}{\sigma-1}}\right], \tag{66}$$

which can be re-written as:

$$-\frac{1}{\sigma-1} \ln \left[\sum_{k\in\Omega_{t,t-1}} S^*_{kt-1} \left(\frac{P_{kt-1}}{P_{kt}}\right)^{\sigma-1}\right] = \ln\left[\frac{\tilde{P}^*_t}{\tilde{P}^*_{t-1}}\right] + \frac{1}{\sigma-1} \ln\left[\left(\frac{\tilde{S}^*_t}{\tilde{S}^*_{t-1}}\right)\right],$$

$$\Lambda^F_t = \Lambda^D_t,$$

$$\Lambda^F_t = -\ln\left[\sum_{k\in\Omega_{t,t-1}} S^*_{kt-1} \left(\frac{P_{kt-1}}{P_{kt}}\right)^{\sigma-1}\right],$$

$$\Lambda^D_t = (\sigma-1)\ln\left[\frac{\tilde{P}^*_t}{\tilde{P}^*_{t-1}}\right] + \ln\left[\frac{\tilde{S}^*_t}{\tilde{S}^*_{t-1}}\right].$$

First, we differentiate Λ^F_t to obtain:

$$\frac{d\Lambda^F_t}{d(\sigma-1)} = -\frac{\sum_{k\in\Omega_{t,t-1}} S^*_{kt-1} \ln\left(\frac{P_{kt-1}}{P_{kt}}\right)\left(\frac{P_{kt-1}}{P_{kt}}\right)^{\sigma-1}}{\sum_{k\in\Omega_{t,t-1}} S^*_{kt-1} \left(\frac{P_{kt-1}}{P_{kt}}\right)^{\sigma-1}}, \tag{67}$$

$$= \frac{\sum_{k\in\Omega_{t,t-1}} S^*_{kt-1} \ln\left(\frac{P_{kt}}{P_{kt-1}}\right)\left(\frac{P_{kt-1}}{P_{kt}}\right)^{\sigma-1}}{\sum_{k\in\Omega_{t,t-1}} S^*_{kt-1} \left(\frac{P_{kt-1}}{P_{kt}}\right)^{\sigma-1}}$$

where we have used $d\left(a^{x}\right)/dx = (\ln a)\, a^{x}$. Now note that the common expenditure share (17), the CES price index for common goods (16) and the time-invariant demand shifters ($\varphi_{ut} = \varphi_{ut-1} = \bar{\varphi}_{u}$) imply:

$$\left(\frac{P_{kt-1}}{P_{kt}}\right)^{\sigma-1} = \frac{S^{*}_{kt}}{S^{*}_{kt-1}} \frac{\left(P^{*}_{t}\right)^{1-\sigma}}{\left(P^{*}_{t-1}\right)^{1-\sigma}}, \qquad k \in \Omega_{t,t-1}. \tag{68}$$

Using this result in (67), re-arranging terms and noting that $\sum_{k\in\Omega_{t,t-1}} S^{*}_{kt-1} = 1$, we obtain:

$$\frac{d\Lambda^{F}_{t}}{d\left(\sigma-1\right)} = \sum_{k\in\Omega_{t,t-1}} S^{*}_{kt} \ln\left(\frac{P_{kt}}{P_{kt-1}}\right). \tag{69}$$

Note that $S^{*}_{kt} > 0$; $\ln\left(P_{kt}/P_{kt-1}\right) < 0$ for $P_{kt} < P_{kt-1}$; and $\ln\left(P_{kt}/P_{kt-1}\right) > 0$ for $P_{kt} > P_{kt-1}$. Therefore, depending on the values of the expenditure shares $\{S^{*}_{kt}\}$, $\frac{d\Lambda^{F}_{t}}{d(\sigma-1)}$ can be either positive or negative, and is independent of $\sigma - 1$. Second, we differentiate Λ^{D}_{t} to obtain:

$$\frac{d\Lambda^{D}_{t}}{d\left(\sigma-1\right)} = \ln\left[\frac{\tilde{P}^{*}_{t}}{\tilde{P}^{*}_{t-1}}\right].$$

Note that $\frac{d\Lambda^{D}_{t}}{d(\sigma-1)} > 0$ for $\tilde{P}^{*}_{t} > \tilde{P}^{*}_{t-1}$ and $\frac{d\Lambda^{D}_{t}}{d(\sigma-1)} < 0$ for $\tilde{P}^{*}_{t} < \tilde{P}^{*}_{t-1}$. Therefore, depending on the values of prices $\{P_{kt}\}$, $\frac{d\Lambda^{D}_{t}}{d(\sigma-1)}$ can be either positive or negative, and is independent of $\sigma - 1$. both $\frac{d\Lambda^{F}_{t}}{d(\sigma-1)}$ and $\frac{d\Lambda^{D}_{t}}{d(\sigma-1)}$ can be either positive or negative and are independent of $\sigma - 1$. Under our assumption that:

$$\sum_{k\in\Omega_{t,t-1}} S^{*}_{kt} \ln\left(\frac{P_{kt}}{P_{kt-1}}\right) \neq \sum_{k\in\Omega_{t,t-1}} \frac{1}{N_{t,t-1}} \ln\left(\frac{P_{kt}}{P_{kt-1}}\right) = \ln\left[\frac{\tilde{P}^{*}_{t}}{\tilde{P}^{*}_{t-1}}\right],$$

we have:

$$\frac{d\Lambda^{F}_{t}}{d\left(\sigma-1\right)} \neq \frac{d\Lambda^{D}_{t}}{d\left(\sigma-1\right)}. \tag{70}$$

Note that $(\sigma - 1)$ can take arbitrarily large negative values ($(\sigma - 1) \to -\infty$) or arbitrarily large positive values ($(\sigma - 1) \to \infty$). Furthermore, the derivatives in (70) differ from one another and are independent of $(\sigma - 1)$. Therefore Λ^{F}_{t} and Λ^{D}_{t} must exhibit a single-crossing property such that there exists a unique value of $(\sigma - 1) \in (-\infty, \infty)$ that satisfies (66), as shown in Figure 10. Having determined σ, we can recover the demand shifters from the expenditure shares (3) for each good $\ell \in \Omega_{t}$, which imply:

$$\ln \varphi_{\ell t} = \ln P_{\ell t} - \ln P_{t} + \frac{1}{\sigma - 1} \ln S_{\ell t}, \qquad \ell \in \Omega_{t}. \tag{71}$$

Taking means of both sides of the equation for goods that are common to both periods, we have:

$$\frac{1}{N_{t,t-1}} \sum_{k\in\Omega_{t,t-1}} \ln \varphi_{kt} = \frac{1}{N_{t,t-1}} \sum_{k\in\Omega_{t,t-1}} \ln P_{kt} - \ln P_{t} + \frac{1}{\sigma-1}\frac{1}{N_{t,t-1}} \sum_{k\in\Omega_{t,t-1}} \ln S_{kt}, \qquad k \in \Omega_{t,t-1}. \tag{72}$$

Taking differences between (71) and (72), and exponentiating, we obtain:

$$\frac{\varphi_{\ell t}}{\tilde{\varphi}^{*}_{t}} = \frac{P_{\ell t}}{\tilde{P}^{*}_{t}} \left(\frac{S_{\ell t}}{\tilde{S}^{*}_{t}}\right)^{\frac{1}{\sigma-1}}, \tag{73}$$

where a tilde above a variable denotes the geometric mean of that variable and the asterisks indicate that this geometric mean is taken across the set of common goods ($\Omega_{t,t-1}$). Note that the right-hand side of (73) can be computed from observed prices and expenditure shares (P_{kt}, S_{kt}) and determines φ_{kt} up to a normalization that corresponds to a choice of units in which to measure the demand shifters (we impose the normalization $\tilde{\varphi}_t^* = 1$).

Under the identifying assumption of constant aggregate CES preferences (using equations (25) and (26) and $\Theta_{t,t-1}^{-1/(1-\sigma)} = 1$), we also have:

$$\left[\sum_{k\in\Omega_{t,t-1}} S_{kt}^* \left(\frac{P_{kt}}{P_{kt-1}}\right)^{\sigma-1}\right]^{\frac{1}{\sigma-1}} = \left[\frac{\tilde{P}_t^*}{\tilde{P}_{t-1}^*}\left(\frac{\tilde{S}_t^*}{\tilde{S}_{t-1}^*}\right)^{\frac{1}{\sigma-1}}\right], \tag{74}$$

which can be re-written as:

$$\frac{1}{\sigma-1}\ln\left[\sum_{k\in\Omega_{t,t-1}} S_{kt}^* \left(\frac{P_{kt}}{P_{kt-1}}\right)^{\sigma-1}\right] = \ln\left[\frac{\tilde{P}_t^*}{\tilde{P}_{t-1}^*}\right] + \frac{1}{\sigma-1}\ln\left[\left(\frac{\tilde{S}_t^*}{\tilde{S}_{t-1}^*}\right)\right],$$

$$\Lambda_t^B = \Lambda_t^D,$$

$$\Lambda_t^B = \ln\left[\sum_{k\in\Omega_{t,t-1}} S_{kt}^* \left(\frac{P_{kt}}{P_{kt-1}}\right)^{\sigma-1}\right],$$

$$\Lambda_t^D = (\sigma-1)\ln\left[\frac{\tilde{P}_t^*}{\tilde{P}_{t-1}^*}\right] + \ln\left[\frac{\tilde{S}_t^*}{\tilde{S}_{t-1}^*}\right].$$

First, we differentiate Λ_t^B to obtain:

$$\frac{d\Lambda_t^B}{d(\sigma-1)} = \frac{\sum_{k\in\Omega_{t,t-1}} S_{kt}^* \ln\left(\frac{P_{kt}}{P_{kt-1}}\right)\left(\frac{P_{kt}}{P_{kt-1}}\right)^{\sigma-1}}{\sum_{k\in\Omega_{t,t-1}} S_{kt}^* \left(\frac{P_{kt}}{P_{kt-1}}\right)^{\sigma-1}}, \tag{75}$$

where we have again used $d(a^x)/dx = (\ln a)\,a^x$. Using the relationship between relative prices and relative expenditure shares for common goods (68), and re-arranging terms and noting that $\sum_{k\in\Omega_{t,t-1}} S_{kt}^* = 1$, we obtain:

$$\frac{d\Lambda_t^B}{d(\sigma-1)} = \sum_{k\in\Omega_{t,t-1}} S_{kt-1}^* \ln\left(\frac{P_{kt}}{P_{kt-1}}\right). \tag{76}$$

Note that $S_{kt-1}^* > 0$; $\ln(P_{kt}/P_{kt-1}) < 0$ for $P_{kt} < P_{kt-1}$; and $\ln(P_{kt}/P_{kt-1}) > 0$ for $P_{kt} > P_{kt-1}$. Therefore, depending on the values of the expenditure shares $\{S_{kt-1}^*\}$, $\frac{d\Lambda_t^B}{d(\sigma-1)}$ can be either positive or negative, and is independent of $\sigma-1$. Second, we differentiate Λ_t^D to obtain:

$$\frac{d\Lambda_t^D}{d(\sigma-1)} = \ln\left[\frac{\tilde{P}_t^*}{\tilde{P}_{t-1}^*}\right].$$

Therefore both $\frac{d\Lambda_t^B}{d(\sigma-1)}$ and $\frac{d\Lambda_t^D}{d(\sigma-1)}$ can be either positive or negative and are independent of $\sigma-1$. Under our assumption that:

$$\sum_{k\in\Omega_{t,t-1}} S_{kt-1}^* \ln\left(\frac{P_{kt}}{P_{kt-1}}\right) \neq \sum_{k\in\Omega_{t,t-1}} \frac{1}{N_{t,t-1}} \ln\left(\frac{P_{kt}}{P_{kt-1}}\right) = \ln\left[\frac{\tilde{P}_t^*}{\tilde{P}_{t-1}^*}\right],$$

we have:

$$\frac{d\Lambda_t^B}{d(\sigma-1)} \neq \frac{d\Lambda_t^D}{d(\sigma-1)}. \tag{77}$$

Note that $(\sigma-1)$ can take arbitrarily large negative values ($(\sigma-1) \to -\infty$) or arbitrarily large positive values ($(\sigma-1) \to \infty$). Furthermore, the derivatives in (77) differ from one another and are independent of $(\sigma-1)$. Therefore Λ_t^B and Λ_t^D must exhibit a single-crossing property such that there exists a unique value of $(\sigma-1) \in (-\infty,\infty)$ that satisfies (74), as shown in Figure 10. Having determined σ, we can recover the demand shifters from the expenditure shares (3) for each good $\ell \in \Omega_t$, as shown above. When $\Theta_{t-1,t}^{1/(1-\sigma)} = \Theta_{t,t-1}^{-1/(1-\sigma)} = 1$, Λ_t^F and Λ_t^D cross at the same value of σ as Λ_t^B and Λ_t^D, which corresponds to the single-crossing point for a consumer with time-invariant preferences for each good (since $\varphi_{kt} = \overline{\varphi}_k$ implies $\Theta_{t-1,t}^{1/(1-\sigma)} = \Theta_{t,t-1}^{-1/(1-\sigma)} = 1$ in equation (27)), as shown in Figure 10.

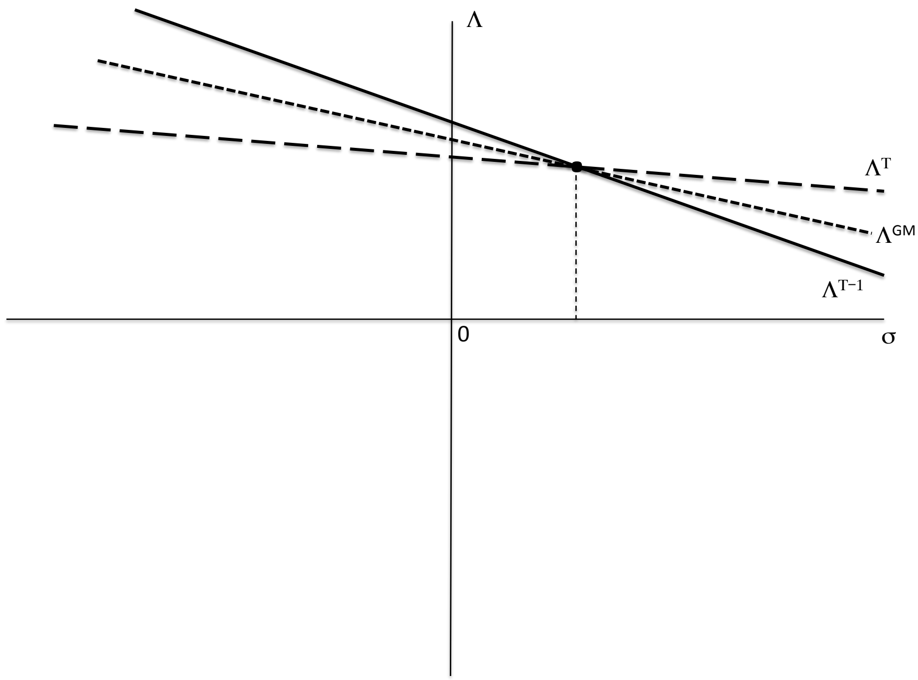

Figure 10: Single crossing between Λ^{T-1} , Λ^T and Λ^{GM}

□

7.4 Proof of Proposition 3 (Consistency for Small Demand Shocks)

Proof. Recall that the moment function is:

$$m_t(\sigma) = \begin{pmatrix} \ln\left[\sum_{k\in\Omega_{t,t-1}} S_{kt-1}^* \left(\frac{P_{kt}}{P_{kt-1}}\right)^{1-\sigma}\right] - (1-\sigma)\ln\left[\frac{\tilde{P}_t^*}{\tilde{P}_{t-1}^*}\left(\frac{\tilde{S}_t^*}{\tilde{S}_{t-1}^*}\right)^{\frac{1}{\sigma-1}}\right] \\ -\ln\left[\sum_{k\in\Omega_{t,t-1}} S_{kt}^* \left(\frac{P_{kt}}{P_{kt-1}}\right)^{-(1-\sigma)}\right] - (1-\sigma)\ln\left[\frac{\tilde{P}_t^*}{\tilde{P}_{t-1}^*}\left(\frac{\tilde{S}_t^*}{\tilde{S}_{t-1}^*}\right)^{\frac{1}{\sigma-1}}\right] \end{pmatrix} = \begin{pmatrix} -\ln\left(\Theta_{t-1,t}\right) \\ \ln\left(\Theta_{t,t-1}\right) \end{pmatrix}, \tag{78}$$

where

$$\Theta_{t-1,t} = \frac{1}{\sum_{k\in\Omega_{t,t-1}} S_{kt}^* \left(\frac{\varphi_{kt-1}}{\varphi_{kt}}\right)^{\sigma-1}}, \qquad \Theta_{t,t-1} = \frac{1}{\sum_{k\in\Omega_{t,t-1}} S_{kt-1}^* \left(\frac{\varphi_{kt}}{\varphi_{kt-1}}\right)^{\sigma-1}}.$$

As demand shocks become small ($\varphi_{kt}/\varphi_{kt-1} \to 1$ for all k), both aggregate demand shifters converge in distribution to one:

$$\Theta_{t-1,t} \xrightarrow{d} 1, \qquad \Theta_{t,t-1} \xrightarrow{d} 1,$$

which implies that the moment function converges in distribution to zero:

$$m_t(\sigma) \xrightarrow{d} \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Therefore the moment function (78) reduces to:

$$\left[\sum_{k\in\Omega_{t,t-1}} S^*_{kt-1}\left(\frac{P_{kt}}{P_{kt-1}}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}} = \frac{\tilde{P}^*_t}{\tilde{P}^*_{t-1}}\left(\frac{\tilde{S}^*_t}{\tilde{S}^*_{t-1}}\right)^{\frac{1}{\sigma-1}}, \tag{79}$$

$$\left[\sum_{k\in\Omega_{t,t-1}} S^*_{kt}\left(\frac{P_{kt}}{P_{kt-1}}\right)^{-(1-\sigma)}\right]^{-\frac{1}{1-\sigma}} = \frac{\tilde{P}^*_t}{\tilde{P}^*_{t-1}}\left(\frac{\tilde{S}^*_t}{\tilde{S}^*_{t-1}}\right)^{\frac{1}{\sigma-1}}. \tag{80}$$

From the proof of Proposition 2, the equalities (79)-(80) identify the true value of the elasticity of substitution (σ). Using this true value for the elasticity of substitution (σ), the expenditure share (3) identifies the true value of demand for each good k and time period t (φ_{kt}). □

7.5 Proof of Proposition 4 (Consistency for Large Number Common Goods)

Proof. Recall that the moment function is:

$$m_t(\sigma) = \begin{pmatrix} \ln\left[\sum_{k\in\Omega_{t,t-1}} S^*_{kt-1}\left(\frac{P_{kt}}{P_{kt-1}}\right)^{1-\sigma}\right] - (1-\sigma)\ln\left[\frac{\tilde{P}^*_t}{\tilde{P}^*_{t-1}}\left(\frac{\tilde{S}^*_t}{\tilde{S}^*_{t-1}}\right)^{\frac{1}{\sigma-1}}\right] \\ -\ln\left[\sum_{k\in\Omega_{t,t-1}} S^*_{kt}\left(\frac{P_{kt}}{P_{kt-1}}\right)^{-(1-\sigma)}\right] - (1-\sigma)\ln\left[\frac{\tilde{P}^*_t}{\tilde{P}^*_{t-1}}\left(\frac{\tilde{S}^*_t}{\tilde{S}^*_{t-1}}\right)^{\frac{1}{\sigma-1}}\right] \end{pmatrix} = \begin{pmatrix} -\ln(\Theta_{t-1,t}) \\ \ln(\Theta_{t,t-1}) \end{pmatrix}, \tag{81}$$

where

$$\Theta_{t-1,t} = \frac{1}{\Xi^B} = \frac{1}{\sum_{k\in\Omega_{t,t-1}} S^*_{kt}\left(\frac{\varphi_{kt-1}}{\varphi_{kt}}\right)^{\sigma-1}}, \tag{82}$$

$$\Theta_{t,t-1} = \frac{1}{\Xi^F} = \frac{1}{\sum_{k\in\Omega_{t,t-1}} S^*_{kt-1}\left(\frac{\varphi_{kt}}{\varphi_{kt-1}}\right)^{\sigma-1}}. \tag{83}$$

Consider the following term:

$$\Xi^B = \sum_{k\in\Omega_{t,t-1}} S^*_{kt}\left(\frac{\varphi_{kt-1}}{\varphi_{kt}}\right)^{\sigma-1} = \sum_{k\in\Omega_{t,t-1}} S^*_{kt}\left(\frac{\varphi_{kt}}{\varphi_{kt-1}}\right)^{-(\sigma-1)}. \tag{84}$$

Assume:

$$\varepsilon_{kt} = \frac{\varphi_{kt}}{\varphi_{kt-1}} \sim \text{i.i.d.}\left(1, \chi^2_\varepsilon\right). \tag{85}$$

Using this assumption in (84), we have:

$$\Xi^B = \sum_{k\in\Omega_{t,t-1}} S^*_{kt}\varepsilon_{kt}^{-(\sigma-1)}. \tag{86}$$

From the Central Limit Theorem, we have:

$$\left[\frac{1}{N_{t,t-1}}\sum_{k\in\Omega_{t,t-1}}(\varepsilon_{kt}-1)\right] \sim \mathcal{N}\left(0, \frac{\chi^2_\varepsilon}{N_{t,t-1}}\right)$$

From the Delta method, a sequence of random variables that satisfies:

$$\sqrt{n}\,(Y_n-\mu) \xrightarrow{d} \mathcal{N}\left(0,\chi^2\right)$$

implies:

$$\sqrt{n}\left[\left[g\left(Y_n\right)-g\left(\mu\right)\right]\xrightarrow{d}\mathcal{N}\left(0,\chi^2\left[g'\left(\mu\right)\right]^2\right)\right].$$

From the above, we have $g\left(\cdot\right)=\left(\cdot\right)^{-(\sigma-1)}$, $\mu=1$, $\chi^2=\chi^2_{\varepsilon}$, $g\left(\mu\right)=1$ and $g'\left(\mu\right)=-\left(\sigma-1\right)$, which implies:

$$\left[\frac{1}{N_{t,t-1}}\sum_{k\in\Omega_{t,t-1}}\left(\varepsilon_{kt}^{-(\sigma-1)}-1\right)\right]\xrightarrow{d}\mathcal{N}\left(0,\frac{\chi^2_{\varepsilon}}{N_{t,t-1}}\left(\sigma-1\right)^2\right).$$

Therefore, as $N_{t,t-1}\rightarrow\infty$, we have:

$$\left[\frac{1}{N_{t,t-1}}\sum_{k\in\Omega_{t,t-1}}\left(\varepsilon_{kt}^{-(\sigma-1)}-1\right)\right]\xrightarrow{d}0. \tag{87}$$

Now returning to (86), we have:

$$\begin{aligned}\Xi^B &= \left[\sum_{k\in\Omega_{t,t-1}}S^*_{kt}\left(\varepsilon_{kt}^{-(\sigma-1)}-1\right)\right]+1. &(88)\\ &=\left\{N_{t,t-1}\left[\frac{1}{N_{t,t-1}}\sum_{k\in\Omega_{t,t-1}}S^*_{kt}\left(\varepsilon_{kt}^{-(\sigma-1)}-1\right)\right]\right\}+1\\ &\leq\left\{N_{t,t-1}S^{*\max}_{N_{t,t-1}t}\left[\frac{1}{N_{t,t-1}}\sum_{k\in\Omega_{t,t-1}}\left(\varepsilon_{kt}^{-(\sigma-1)}-1\right)\right]\right\}+1.\end{aligned}$$

Now note that □

$$\begin{aligned}N_{t,t-1}S^{*\max}_{N_{t,t-1}t}&=\frac{N_{t,t-1}\sup\left\{\left(P_{1t}/\varphi_{1t}\right)^{1-\sigma},\ldots,\left(P_{N_{t,t-1}t}/\varphi_{N_{t,t-1}t}\right)^{1-\sigma}\right\}}{\sum_{k=1}^{N_{t,t-1}}\left(P_{kt}/\varphi_{kt}\right)^{1-\sigma}}, &(89)\\ &=\frac{\sup\left\{\left(P_{1t}/\varphi_{1t}\right)^{1-\sigma},\ldots,\left(P_{N_{t,t-1}t}/\varphi_{N_{t,t-1}t}\right)^{1-\sigma}\right\}}{\frac{1}{N_{t,t-1}}\sum_{k=1}^{N_{t,t-1}}\left(P_{kt}/\varphi_{kt}\right)^{1-\sigma}}<\infty.\end{aligned}$$

Using (87) and (89) in (88), we obtain the result that as $N_{t,t-1}\rightarrow\infty$:

$$\Xi^B\xrightarrow{d}1. \tag{90}$$

Consider the following term:

$$\Xi^F=\sum_{k\in\Omega_{t,t-1}}S^*_{kt-1}\left(\frac{\varphi_{kt}}{\varphi_{kt-1}}\right)^{\sigma-1}. \tag{91}$$

Using the assumption (85) in (91), we have:

$$\Xi^F=\sum_{k\in\Omega_{t,t-1}}S^*_{kt-1}\varepsilon_{kt}^{\sigma-1}. \tag{92}$$

From the Central Limit Theorem, we again have:

$$\left[\frac{1}{N_{t,t-1}}\sum_{k\in\Omega_{t,t-1}}\left(\varepsilon_{kt}-1\right)\right]\sim\mathcal{N}\left(0,\frac{\chi^2_{\varepsilon}}{N_{t,t-1}}\right)$$

From the Delta method, a sequence of random variables that satisfies:

$$\sqrt{n}\left(Y_n-\mu\right)\xrightarrow{d}\mathcal{N}\left(0,\chi^2\right)$$

implies:

$$\sqrt{n}\left[\left[g\left(Y_n\right)-g\left(\mu\right)\right]\xrightarrow{d}\mathcal{N}\left(0,\chi^2\left[g'\left(\mu\right)\right]^2\right)\right].$$

From the above, we have $g\left(\cdot\right)=\left(\cdot\right)^{\sigma-1}$, $\mu=1$, $\chi^2=\chi^2_{\varepsilon}$, $g\left(\mu\right)=1$ and $g'\left(\mu\right)=\sigma-1$, which implies:

$$\left[\frac{1}{N_{t,t-1}}\sum_{k\in\Omega_{t,t-1}}\left(\varepsilon_{kt}^{\sigma-1}-1\right)\right]\xrightarrow{d}\mathcal{N}\left(0,\frac{\chi^2_{\varepsilon}}{N_{t,t-1}}\left(\sigma-1\right)^2\right).$$

Therefore, as $N_{t,t-1}\to\infty$, we have:

$$\left[\frac{1}{N_{t,t-1}}\sum_{k\in\Omega_{t,t-1}}\left(\varepsilon_{kt}^{\sigma-1}-1\right)\right]\xrightarrow{d}0. \tag{93}$$

Now returning to (92), we have:

$$\begin{aligned}\Xi^F&=\left[\sum_{k\in\Omega_{t,t-1}}S^*_{kt-1}\left(\varepsilon_{kt}^{\sigma-1}-1\right)\right]+1. \qquad (94)\\&=\left\{N_{t,t-1}\left[\frac{1}{N_{t,t-1}}\sum_{k\in\Omega_{t,t-1}}S^*_{kt-1}\left(\varepsilon_{kt}^{\sigma-1}-1\right)\right]\right\}+1\\&\leq\left\{N_{t,t-1}S^{*\max}_{N_{t,t-1}t-1}\left[\frac{1}{N_{t,t-1}}\sum_{k\in\Omega_{t,t-1}}\left(\varepsilon_{kt}^{\sigma-1}-1\right)\right]\right\}+1.\end{aligned}$$

Now note that

$$\begin{aligned}N_{t,t-1}S^{*\max}_{N_{t,t-1}t-1}&=\frac{N_{t,t-1}\sup\left\{\left(P_{1t-1}/\varphi_{1t-1}\right)^{1-\sigma},\ldots,\left(P_{N_{t,t-1}t-1}/\varphi_{N_{t,t-1}t-1}\right)^{1-\sigma}\right\}}{\sum_{k=1}^{N_{t,t-1}}\left(P_{kt-1}/\varphi_{kt-1}\right)^{1-\sigma}}, \qquad (95)\\&=\frac{\sup\left\{\left(P_{1t-1}/\varphi_{1t-1}\right)^{1-\sigma},\ldots,\left(P_{N_{t,t-1}t-1}/\varphi_{N_{t,t-1}t-1}\right)^{1-\sigma}\right\}}{\frac{1}{N_{t,t-1}}\sum_{k=1}^{N_{t,t-1}}\left(P_{kt-1}/\varphi_{kt-1}\right)^{1-\sigma}}<\infty.\end{aligned}$$

Using (93) and (95) in (94), we obtain the result that as $N_{t,t-1}\to\infty$:

$$\Xi^F\xrightarrow{d}1. \tag{96}$$

Using (90) and (96) in (82) and (83), we have:

$$\begin{aligned}\Xi^B\xrightarrow{d}1\qquad&\Rightarrow\qquad\Theta_{t-1,t}\xrightarrow{d}1,\\\Xi^F\xrightarrow{d}1\qquad&\Rightarrow\qquad\Theta_{t,t-1}\xrightarrow{d}1,\end{aligned}$$

which implies that the moment function (81) converges in distribution to zero:

$$m_t\left(\sigma\right)\xrightarrow{d}\begin{pmatrix}0\\0\end{pmatrix}.$$

Therefore the moment function (81) reduces to:

$$\left[\sum_{k\in\Omega_{t,t-1}} S^*_{kt-1}\left(\frac{P_{kt}}{P_{kt-1}}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}} = \frac{\tilde{P}^*_t}{\tilde{P}^*_{t-1}}\left(\frac{\tilde{S}^*_t}{\tilde{S}^*_{t-1}}\right)^{\frac{1}{\sigma-1}}, \tag{97}$$

$$\left[\sum_{k\in\Omega_{t,t-1}} S^*_{kt}\left(\frac{P_{kt}}{P_{kt-1}}\right)^{-(1-\sigma)}\right]^{-\frac{1}{1-\sigma}} = \frac{\tilde{P}^*_t}{\tilde{P}^*_{t-1}}\left(\frac{\tilde{S}^*_t}{\tilde{S}^*_{t-1}}\right)^{\frac{1}{\sigma-1}}. \tag{98}$$

From the proof of Proposition 2, the equalities (97)-(98) identify the true value of the elasticity of substitution (σ). Using this true value for the elasticity of substitution (σ), the expenditure share (3) identifies the true value of demand for each good k and time period t (φ_{kt}).

7.6 Proof of Proposition 6

(a) Under the assumption of constant demand for each good ($\varphi_{kt} = \varphi_{kt-1} = \bar{\varphi}_k$ for all k and t), the common goods expenditure share is:

$$S^*_{kt} = \frac{\left(P_{kt}/\varphi_k\right)^{1-\sigma}}{\sum_{\ell\in\Omega_{t,t-1}}\left(P_{\ell t}/\varphi_\ell\right)^{1-\sigma}}. \tag{99}$$

Dividing the expenditure share by its geometric mean, we get:

$$\frac{S^*_{kt}}{\tilde{S}^*_t} = \left(\frac{P_{kt}/\bar{\varphi}_k}{\tilde{P}_t}\right)^{1-\sigma}, \tag{100}$$

where we have used our normalization that $\tilde{\varphi} = 1$. Taking logarithms in (100) we obtain the following equation

$$\ln\left(\frac{S^*_{kt}}{\tilde{S}^*_t}\right) = (1-\sigma)\ln\left(\frac{P_{kt}}{\tilde{P}_t}\right) + (\sigma-1)\ln\bar{\varphi}_k. \tag{101}$$

Taking differences in (101), we obtain:

$$\Delta\ln\left(\frac{S^*_{kt}}{\tilde{S}^*_t}\right) = (1-\sigma)\,\Delta\ln\left(\frac{P_{kt}}{\tilde{P}_t}\right). \tag{102}$$

Multiplying both sides of (102) by ω^*_{kt} and summing across common goods, we get:

$$\sum_{k\in\Omega_{t,t-1}}\omega^*_{kt}\Delta\ln\left(\frac{S^*_{kt}}{\tilde{S}^*_t}\right) = (1-\sigma)\sum_{k\in\Omega_{t,t-1}}\omega^*_{kt}\Delta\ln\left(\frac{P_{kt}}{\tilde{P}_t}\right), \tag{103}$$

where ω^*_{kt} are the Sato-Vartia weights:

$$\omega^*_{kt} = \frac{\frac{S^*_{kt}-S^*_{kt-1}}{\ln S^*_{kt}-\ln S^*_{kt-1}}}{\sum_{\ell\in\Omega_{t,t-1}}\frac{S^*_{\ell t}-S^*_{\ell t-1}}{\ln S^*_{\ell t}-\ln S^*_{\ell t-1}}}.$$

Note that (103) also be used to solve for σ given observed prices and expenditure shares:

$$\sigma^{SV} = 1 + \frac{\sum_{k\in\Omega_{t,t-1}}\omega^*_{kt}\left[\ln\left(\frac{S^*_{kt}}{S^*_{kt-1}}\right) - \ln\left(\frac{\tilde{S}^*_t}{\tilde{S}^*_{t-1}}\right)\right]}{\sum_{k\in\Omega_{t,t-1}}\omega^*_{kt}\left[\ln\left(\frac{\tilde{P}_t}{\tilde{P}_{t-1}}\right) - \ln\left(\frac{P_{kt}}{P_{kt-1}}\right)\right]}, \tag{104}$$

which establishes that the elasticity of substitution (σ) is uniquely identified from observed changes in prices and expenditure shares with no estimation. Note that we could have instead multiplied both sides of (102) by any finite share that sums to one across common goods:

$$\sum_{k\in\Omega_{t,t-1}}\xi^*_{kt}\Delta\ln\left(\frac{S^*_{kt}}{\tilde{S}^*_t}\right) = (1-\sigma)\sum_{k\in\Omega_{t,t-1}}\xi^*_{kt}\Delta\ln\left(\frac{P_{kt}}{\tilde{P}_t}\right), \qquad \sum_{k\in\Omega_{t,t-1}}\xi^*_{kt} = 1, \tag{105}$$

and obtained another expression for σ given observed prices and expenditure shares:

$$\sigma^{ALT} = 1 + \frac{\sum_{k \in \Omega_{t,t-1}} \xi^*_{kt} \left[\ln\left(\frac{S^*_{kt}}{S^*_{kt-1}}\right) - \ln\left(\frac{\tilde{S}^*_t}{\tilde{S}^*_{t-1}}\right) \right]}{\sum_{k \in \Omega_{t,t-1}} \xi^*_{kt} \left[\ln\left(\frac{\tilde{P}_t}{\tilde{P}_{t-1}}\right) - \ln\left(\frac{P_{kt}}{P_{kt-1}}\right) \right]}. \tag{106}$$

Therefore that there exists a continuum of approaches to measuring σ, each of which weights prices and expenditure shares with different non-negative weights that sum to one. Under the assumption of constant demand for each good ($\varphi_{kt} = \varphi_{kt-1} = \bar{\varphi}_k$ for all k and t), each of these alternative approaches returns the same value for σ, since all are derived from (102).

(b) Suppose that demand for goods changes over time ($\varphi_{kt} \neq \varphi_{kt-1}$ for some k and t), but a researcher falsely assumes that demand for each good is constant. Dividing the common goods expenditure share by its geometric mean, we get:

$$\frac{S^*_{kt}}{\tilde{S}^*_t} = \left(\frac{P_{kt}/\varphi_{kt}}{\tilde{P}_t}\right)^{1-\sigma}, \tag{107}$$

where we have used our normalization of $\tilde{\varphi}_t = 1$. Taking logarithms in (107) and taking differences, we obtain:

$$\Delta \ln\left(\frac{S^*_{kt}}{\tilde{S}^*_t}\right) = (1-\sigma)\,\Delta \ln\left(\frac{P_{kt}}{\tilde{P}_t}\right) + (\sigma - 1)\,\Delta \ln \varphi_{kt}. \tag{108}$$

Multiplying both sides of (108) by ω^*_{kt} and summing across common goods, we get:

$$\sum_{k \in \Omega_{t,t-1}} \omega^*_{kt} \Delta \ln\left(\frac{S^*_{kt}}{\tilde{S}^*_t}\right) = (1-\sigma) \sum_{k \in \Omega_{t,t-1}} \omega^*_{kt} \Delta \ln\left(\frac{P_{kt}}{\tilde{P}_t}\right) + (\sigma - 1) \sum_{k \in \Omega_{t,t-1}} \omega^*_{kt} \Delta \ln \varphi_{kt}, \tag{109}$$

Rearranging (109), we obtain:

$$\sigma_{\varphi,\omega^*} = 1 + \frac{\sum_{k \in \Omega_{t,t-1}} \omega^*_{kt} \left[\ln\left(\frac{S^*_{kt}}{S^*_{kt-1}}\right) - \ln\left(\frac{\tilde{S}^*_t}{\tilde{S}^*_{t-1}}\right) \right]}{\sum_{k \in \Omega_{t,t-1}} \omega^*_{kt} \left[\ln\left(\frac{\tilde{P}_t}{\tilde{P}_{t-1}}\right) - \ln\left(\frac{P_{kt}}{P_{kt-1}}\right) + \ln\left(\frac{\varphi_{kt}}{\varphi_{kt-1}}\right) \right]}, \tag{110}$$

Note that we could have instead multiplied both sides of (108) by any finite share that sums to one across common goods:

$$\sum_{k \in \Omega_{t,t-1}} \xi^*_{kt} \Delta \ln\left(\frac{S^*_{kt}}{\tilde{S}^*_t}\right) = (1-\sigma) \sum_{k \in \Omega_{t,t-1}} \xi^*_{kt} \Delta \ln\left(\frac{P_{kt}}{\tilde{P}_t}\right) + (\sigma - 1) \sum_{k \in \Omega_{t,t-1}} \xi^*_{kt} \Delta \ln \varphi_{kt}, \tag{111}$$

$$\sum_{k \in \Omega_{t,t-1}} \xi^*_{kt} = 1,$$

and obtained another expression for σ given observed prices and expenditure shares:

$$\sigma_{\varphi,\xi} = 1 + \frac{\sum_{k \in \Omega_{t,t-1}} \xi^*_{kt} \left[\ln\left(\frac{S^*_{kt}}{S^*_{kt-1}}\right) - \ln\left(\frac{\tilde{S}^*_t}{\tilde{S}^*_{t-1}}\right) \right]}{\sum_{k \in \Omega_{t,t-1}} \xi^*_{kt} \left[\ln\left(\frac{\tilde{P}_t}{\tilde{P}_{t-1}}\right) - \ln\left(\frac{P_{kt}}{P_{kt-1}}\right) + \ln\left(\frac{\varphi_{kt}}{\varphi_{kt-1}}\right) \right]}. \tag{112}$$

Note that both of the approaches (110) and (112) return the same value for σ, because both are derived from (108). However, suppose that a researcher falsely assumes that demand for each good is constant ($\varphi_{kt} = \varphi_{kt-1} = \bar{\varphi}_k$ for all k and t) and uses (104) and (106) to measure σ (instead of (110) and (112)). Under this false assumption, (104) and (106) will return different values for σ, because in general:

$$\sum_{k \in \Omega_{t,t-1}} \omega^*_{kt} \ln(\varphi_{kt}/\varphi_{kt-1}) \neq \sum_{k \in \Omega_{t,t-1}} \xi^*_{kt} \ln(\varphi_{kt}/\varphi_{kt-1}) \qquad \text{when} \qquad \omega^*_{kt} \neq \xi^*_{kt}.$$

Therefore, when demand for goods changes over time ($\varphi_{kt} \neq \varphi_{kt-1}$ for some k and t) but a researcher falsely assumes that demand for each good is constant ($\varphi_{kt} = \varphi_{kt-1} = \bar{\varphi}_k$ for all k and t), the use of different weights for prices and expenditure shares (ω^*_{kt} versus ξ^*_{kt}) returns different elasticities of substitution in general ($\sigma^{SV} \neq \sigma^{ALT}$). Note that our normalization of $\tilde{\varphi}_t = 1$ implies that on average changes in demand for goods are zero:

$$\tilde{\varphi}_t = 1 \qquad \Leftrightarrow \qquad \frac{1}{N_{t,t-1}} \sum_{k \in \Omega_{t,t-1}} \Delta \ln \varphi_{kt} = 0.$$

Therefore, even assuming that on average changes in demand for goods are zero, the use of different weights for prices and expenditure shares (ω^*_{kt} versus ξ^*_{kt}) returns different elasticities of substitution in general ($\sigma^{SV} \neq \sigma^{ALT}$) if demand for goods changes over time but a researcher falsely assumes that demand for each good is constant.

7.7 Proof of Proposition 7

From the common goods expenditure share (17), we can express the change in the common goods price index as:

$$\frac{\mathbb{P}^*_t}{\mathbb{P}^*_{t-1}} = \frac{\left(P_{kt} / \hat{\varphi}^{RW}_{kt}\right) / \left(P_{kt-1} / \hat{\varphi}^{RW}_{kt-1}\right)}{\left(S^*_{kt} / S^*_{kt-1}\right)^{\frac{1}{1-\hat{\sigma}^{RW}}}} \tag{113}$$

Taking logs of both sides, and rearranging, produces:

$$\frac{\ln \frac{\mathbb{P}^*_t}{\mathbb{P}^*_{t-1}} - \ln \frac{P_{kt} / \hat{\varphi}^{RW}_{kt}}{P_{kt-1} / \hat{\varphi}^{RW}_{kt-1}}}{\ln \frac{S^*_{kt}}{S^*_{kt-1}}} = \frac{1}{\hat{\sigma}^{RW} - 1}. \tag{114}$$

If we now multiply both sides of this equation by $S^*_{kt} - S^*_{kt-1}$ and sum across all common goods, we obtain:

$$\sum_{k \in \Omega_{t,t-1}} \left(S^*_{kt} - S^*_{kt-1}\right) \frac{\ln \frac{\mathbb{P}^*_t}{\mathbb{P}^*_{t-1}} - \ln \frac{P_{kt} / \hat{\varphi}^{RW}_{kt}}{P_{kt-1} / \hat{\varphi}^{RW}_{kt-1}}}{\ln \frac{S^*_{kt}}{S^*_{kt-1}}} = 0 \tag{115}$$

or

$$\sum_{k \in \Omega_{t,t-1}} \frac{\left(S^*_{kt} - S^*_{kt-1}\right)}{\ln S^*_{kt} - \ln S^*_{kt-1}} \ln \frac{\mathbb{P}^*_t}{\mathbb{P}^*_{t-1}} = \sum_{k \in \Omega_{t,t-1}} \frac{\left(S^*_{kt} - S^*_{kt-1}\right)}{\ln S^*_{kt} - \ln S^*_{kt-1}} \ln \frac{P_{kt} / \hat{\varphi}^{RW}_{kt}}{P_{kt-1} / \hat{\varphi}^{RW}_{kt-1}}. \tag{116}$$

Taking exponents of both sides of this equation, we obtain our expression for the exact CES price index for common goods with time-varying product appeal:

$$\frac{\mathbb{P}^*_t}{\mathbb{P}^*_{t-1}} = \prod_{k \in \Omega_{t,t-1}} \left(\frac{P_{kt} / \hat{\varphi}^{RW}_{kt}}{P_{kt-1} / \hat{\varphi}^{RW}_{kt-1}}\right)^{\omega^*_{kt}}, \tag{117}$$

where ω^*_{kt} is defined as:

$$\hat{\varphi}_{kt}^{RW} = \frac{P_{kt}}{\tilde{P}_t^*}\left(\frac{S_{kt}}{\tilde{S}_t^*}\right)^{\frac{1}{\hat{\sigma}^{RW}-1}}, \qquad \omega_{kt}^* = \frac{\frac{S_{kt}^*-S_{kt-1}^*}{\ln S_{kt}^*-\ln S_{kt-1}^*}}{\sum_{\ell\in\Omega_{t,t-1}}\frac{S_{\ell t}^*-S_{\ell t-1}^*}{\ln S_{\ell t}^*-\ln S_{\ell t-1}^*}}, \qquad \sum_{k\in\Omega_{t,t-1}}\omega_{kt}^* = 1. \tag{118}$$

To determine the exact CES price index (117), we require observed prices and expenditure shares (P_{kt}, S_{kt}), but also the reverse-weighting estimates of demand for each good ($\hat{\varphi}_{kt}^{RW}$) and the elasticity of substitution (σ^{RW}). In Propositions 2-4, we showed how these reverse-weighting estimates are determined from the observed price and expenditure share data (P_{kt}, S_{kt}).

7.8 Proof of Proposition 8

Proof. From (35), the log change in the exact common goods can be written as:

$$\ln\Phi_t = \ln\Phi_t^{SV} - N_{t,t-1}\left[\frac{1}{N_{t,t-1}}\sum_{k\in\Omega_{t,t-1}}\omega_{kt}^*\ln\left(\frac{\hat{\varphi}_{kt}^{RW}}{\hat{\varphi}_{kt-1}^{RW}}\right)\right],$$

and hence:

$$\ln\Phi_t = \ln\Phi_t^{SV} - N_{t,t-1}\mathbb{E}\left[\omega_{kt}^*\ln\left(\frac{\hat{\varphi}_{kt}^{RW}}{\hat{\varphi}_{kt-1}^{RW}}\right)\right].$$

It follows that:

$$\mathbb{E}\left[\omega_{kt}^*\ln\left(\frac{\hat{\varphi}_{kt}^{RW}}{\hat{\varphi}_{kt-1}^{RW}}\right)\right] > 0 \Rightarrow \ln\Phi_t^{SV} > \ln\Phi_t,$$
$$\mathbb{E}\left[\omega_{kt}^*\ln\left(\frac{\hat{\varphi}_{kt}^{RW}}{\hat{\varphi}_{kt-1}^{RW}}\right)\right] < 0 \Rightarrow \ln\Phi_t^{SV} < \ln\Phi_t,$$
$$\mathbb{E}\left[\omega_{kt}^*\ln\left(\frac{\hat{\varphi}_{kt}^{RW}}{\hat{\varphi}_{kt-1}^{RW}}\right)\right] = 0 \Rightarrow \ln\Phi_t^{SV} = \ln\Phi_t.$$

□

7.9 Translog Specification

While we focus on CES utility, the same conceptual points apply in more flexible functional forms. Consider the following homothetic translog common goods price index in which we allow for time-varying demand for each good (φ_{kt}):

$$\ln P_t = \ln\alpha_0 + \sum_{k\in\Omega_{t,t-1}}\alpha_k\ln\left(\frac{P_{kt}}{\varphi_{kt}}\right) + \frac{1}{2}\sum_{k\in\Omega_{t,t-1}}\sum_{\ell\in\Omega_{t,t-1}}\beta_{k\ell}\ln\left(\frac{P_{kt}}{\varphi_{kt}}\right)\ln\left(\frac{P_{\ell t}}{\varphi_{\ell t}}\right), \tag{119}$$

where we implicitly impose a normalization for demand for each good. The implied common good expenditure share is:

$$S_{\ell t}^* = \alpha_\ell + \sum_{k\in\Omega_{t,t-1}}\beta_{\ell k}\ln\left(\frac{P_{kt}}{\varphi_{kt}}\right). \tag{120}$$

The implied change in the cost of living from period $t-1$ to period t can be expressed as the following index number:

$$\Phi_{t-1,t} = \prod_{k\in\Omega_{t,t-1}}\left(\frac{P_{kt}/\varphi_{kt}}{P_{kt-1}/\varphi_{kt-1}}\right)^{\frac{1}{2}\left(S_{kt}^*+S_{kt-1}^*\right)}. \tag{121}$$

The standard Törnqvist index number for the change in the cost of living from period $t-1$ to period t is:

$$\Phi_{t-1,t} = \prod_{k \in \Omega_{t,t-1}} \left(\frac{P_{kt}}{P_{kt-1}} \right)^{\frac{1}{2}\left(S^{*}_{kt}+S^{*}_{kt-1}\right)}, \tag{122}$$

which is only exact for the translog functional form under the assumption of constant demand for each good ($\varphi_{kt}/\varphi_{kt-1} = 1$).

Therefore, even with a flexible functional form such as translog, standard index numbers in (122) assume time-invariant demand for each good to ensure constant aggregate utility and time reversibility. In contrast, demand systems estimation with such flexible functional forms typically requires a structural residual in the form of time-varying demand for each good in order to rationalize the observed data on expenditure shares in (120). Hence existing empirical research using flexible functional forms is again inconsistent with the joint implications of the three principles of duality, constant aggregate utility and aggregation. Assuming constant demand for each good ensures constant aggregate utility (time reversibility) but is in general inconsistent with the implications of duality for demand systems estimation where time-varying demand for each good is required to rationalize the data.